(*vi*) Point estimators may be more useful than interval estimators because probability statements are attached to point estimates. (*false*)

(*vii*) The point estimation provides two values with a probability statement for estimating the unknown true value of a population parameter. (*false*)

(*viii*) A confidence interval is a type of statistical inference. (*true*)

(*ix*) A point estimate provides information about the precision of the estimate. (*false*)

4. Mark off the following statements as *true* or *false*.

(*i*) We cannot control the precision of an interval estimate by the choice of sample size or level of confidence. (*false*)

(*ii*) The width of a confidence interval increases if the confidence coefficient is decreased. (*false*)

(*iii*) The width of a confidence interval decreases if the confidence coefficient is decreased. (*true*)

(*iv*) The width of a confidence interval can be decreased by decreasing the confidence coefficient. (*true*)

(*v*) The precision of an interval estimate can be increased by decreasing the sample size. (*false*)

(*vi*) The precision of an interval estimate can be increased either by increasing the sample size or by decreasing the confidence coefficient. (*true*)

(*vii*) α is the probability that the interval estimator includes the unknown true value of the population parameter. (*false*)

(*viii*) The statistic T can be used in making confidence interval for μ when population is non-normal. (*false*)

13 HYPOTHESIS TESTING

13.1 THE ELEMENTS OF A TEST OF HYPOTHESIS

We are often concerned with testing or drawing a conclusion about the population parameter θ based on a simple random sample data. In this chapter we present formal structures for making inferences about population parameters such as μ, π, σ^2, *etc.*, we will begin by introducing a *test of hypothesis*.

13.1.1 Statistical Hypothesis. A *statistical hypothesis* is an assertion or conjecture about the distribution of one or more random variables. It is an assertion about the nature of a population. This assertion may or may not be true. Its validity is tested on the basis of an observed random sample. Hypotheses are usually phrased quantitatively in terms of population parameters. The following are some examples of hypotheses.

(*i*) $\mu = 25$ (A population mean equals 25)

(*ii*) $\mu < 40$ (A population mean is less than 40)

(*iii*) $\mu \geq 20$ (A population mean is greater than or equal to 20)

(*iv*) $\pi = 0.4$ (A population proportion equals 0.4)

13.1.2 Hypothesis Testing. The *statistical hypothesis testing* is a procedure to determine whether or not an assumption about some parameter of a population is supported by the information obtained from the observed random sample.

13.1.3 Specification of the Form of Population Distribution. The experimenter, or researcher, must make an assumption about the nature of the underlying distribution of the population. Is the random variable normal or binomial. Or does it follow any other form of the distribution. It is, therefore, necessary to identify the theoretical probability distribution of the random variable under consideration because the decision about the hypothesis is made on the basis of probability of occurrence.

We will find that there are certain elements common to all tests of hypotheses. These elements are introduced and discussed below:

13.1.4 Null Hypothesis. A *null hypothesis*, denoted by H_0, is that hypothesis which is tested for possible rejection (or nullification) under the assumption that it is true.

A null hypothesis is always a statement of either (1) "no effect" or (2) "the status quo", such as the given coin is unbiased, or a drug is ineffective in curing a particular disease, or there is no difference between the two production methods, or the production line requires no preventive maintenance, *etc.* The experiment is conducted to see if this hypothesis is unreasonable.

13.1.5 Establishment of the Null Hypothesis. Let θ represent the true but unknown value of the population parameter and θ_0 a value on the number line, the hypothesis to be tested will take on one of the following three forms.

(*i*) $\theta = \theta_0$, that is, the true value of the population parameter is equal to some specified value θ_0.

(*ii*) $\theta \geq \theta_0$, that is, the true value of the population parameter is equal to or greater than some specified value θ_0.

(*iii*) $\theta \leq \theta_0$ that is, the true value of the population parameter is equal to or less than some specified value θ_0.

13.1.6 Alternative Hypothesis. An *alternative hypothesis*, denoted by H_1, is that hypothesis which we are willing to accept when the null hypothesis is rejected.

An alternative hypothesis gives the opposing conjecture to that given in the null hypothesis. The alternative hypothesis is often called the *research hypothesis,* because this hypothesis expresses the theory that the experimenter, or researcher, believes to be true. The experiment is conducted to see if the alternative hypothesis is supported

13.1.7 Formulation of Null and Alternative Hypothesis. The alternative (or research) hypothesis is a statement about the value of a population parameter that an investigator attempts to support with observed random sample. The statistical hypothesis testing makes use of the null hypothesis that refers to the same population parameter but denies the alternative hypothesis. Thus the basic strategy in statistical hypothesis testing is to attempt to support the research hypothesis by contradicting the null hypothesis. Therefore, when choosing the null and alternative hypotheses, take the following steps:

(*i*) The experiment is conducted to see if there is support for some hypothesis. This will be the alternative hypothesis, expressed as an inequality in the form "less than" or "greater than" or "not equal to".

Example: H_1: $\theta < 40$

(*ii*) State the null hypothesis with an equality sign as a complement of the alternative hypothesis.

Example : H_0: $\theta \geq 40$

The following table presents the three types of alternative hypotheses that constitute the counterparts to the three types of null hypotheses.

	Null hypothesis H_0	Alternative hypothesis H_1
1.	$\theta \geq \theta_0$	$\theta < \theta_0$
2.	$\theta \leq \theta_0$	$\theta > \theta_0$
3.	$\theta = \theta_0$	$\theta \neq \theta_0$ (*i. e.*, $\theta < \theta_0$ or $\theta > \theta_0$)

***Example* 13.1** *Formulate the null and the alternative hypotheses used in test of hypothesis for each of the following*:

(*i*) *The mean lifetime of electric light bulbs newly manufactured by a company has not changed from the previous mean lifetime of* 1200 *hours.*

(*ii*) *An automobile is driven on the average no more than* 16000 *kilometers per year.*

(*iii*) *At least* 10% *of the people of Pakistan pay income tax.*

(*iv*) *The proportion of the households that do not own a colour television set is more than* 0.40 *in a locality.*

(*v*) *The average yield of corn of variety A exceeds the average yield of variety B by at least* 200 *kilogram per acre.*

Solution. The null hypothesis H_0 and the alternative hypothesis H_1 for each of the given situations are:

(*i*)	H_0: μ = 1200 hours	against	H_1: $\mu \neq$ 1200 hours
(*ii*)	H_0: $\mu \leq$ 16000 kilometers	against	H_1: $\mu >$ 16000 kilometers
(*iii*)	H_0: $\pi \geq$ 0.10	against	H_1: $\pi <$ 0.10
(*iv*)	H_0: $\pi \leq$ 0.40	against	H_1: $\pi >$ 0.40
(*v*)	H_0: $\mu_1 - \mu_2 \geq$ 200 kg	against	H_1: $\mu_1 - \mu_2 <$ 200 kg

13.1.8 Simple Hypothesis. A statistical hypothesis is said to be a *simple hypothesis* if it completely specifies the underlying population distribution, namely

(*i*) The functional form of the distribution, and

(*ii*) The specific values of all of its parameters.

That is, if it specifies the particular member of the particular family of probability distributions, *e. g.*, a random variable has a normal distribution with mean $\mu = 30$ and standard deviation $\sigma = 4$; or a random variable has a binomial distribution with $n = 6$ and $\pi = 0.4$.

13.1.9 Composite Hypothesis. A statistical hypothesis is said to be a *composite hypothesis* if it does not completely specify the underlying population distribution, *e. g.*, a random variable has a normal distribution with mean $\mu = 40$ or a random variable has a normal distribution with mean $\mu = 50$ and standard deviation $\sigma \geq 4$; or random variable has a distribution with mean $\mu = 40$ and standard deviation $\sigma = 5$; or a random variable has a binomial distribution with $\pi = 0.4$.

13.1.10 Test Statistic. The *test statistic* is a sample statistic which provides a basis for deciding whether or not the null hypothesis should be rejected. The most commonly used test statistics are Z, T, χ^2, or F.

13.1.11 Rejection Region. A *rejection region* specifies a set of values of the test statistic for which the null hypothesis is rejected (and for which the alternative hypothesis is accepted). It is also called as the *critical region.*

13.1.12 Nonrejection Region. A *nonrejection region* specifies a set of values of the test statistic for which the null hypothesis is not rejected. It is also called as the *noncritical region.*

13.1.13 Critical Values. The values of the test statistic which separate the rejection and nonrejection regions for the test are called *critical values.*

13.1.14 Two-tailed Test. If the critical region is located equally in both tails of the sampling distribution of test statistic, the test is called a *two-tailed* or *two-sided test.* In such tests, the analyst is concerned with detecting values of the test statistic that are either too large or too small to be consistent with the hypothesis being tested.

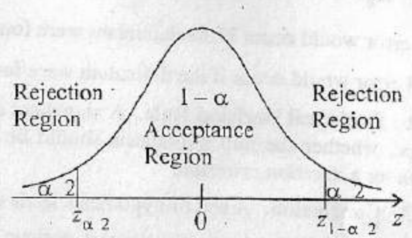

Fig: 13.1 Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

With a two-tailed test, acceptance of null hypothesis means acceptance of a unique value for the population parameter.

13.1.15 One-tailed Test. If the critical region is located in only one tail of the sampling distribution of test statistic, the test is called a *one-tailed* or *one-sided test.*

13.1.16 Left-tailed Test. If the critical region is located in only the left tail of the sampling distribution of test statistic, the test is called a *left-tailed test.* In such tests, the analyst is concerned with detecting values of the test statistic that are too small to be consistent with the hypothesis being tested.

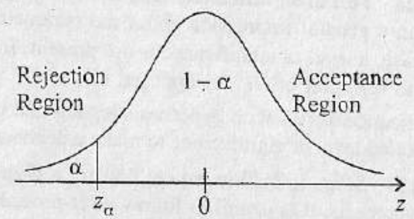

Fig: 13.2 Reject H_0 if $Z < z_\alpha$

With a one-tailed test, acceptance of null hypothesis means accepting that the population parameter is one of many acceptable values.

13.1.17 Right-Tailed Test. If the critical region is located in only the right tail of the sampling distribution of test statistic, the test is called a *right-tailed test.* In such tests the analyst is concerned with detecting the values of the test statistic that are too large to be consistent with the hypothesis being tested.

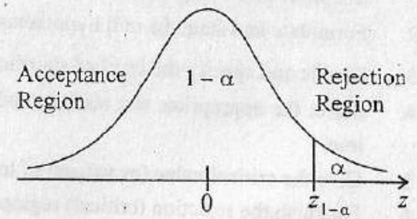

Fig: 13.3 Reject H_0 if $Z > z_{1-\alpha}$

13.1.18 Deciding upon an Appropriate Test. Now the question arises, that how we should decide upon an appropriate test. While deciding upon an appropriate test the following hints are helpful:

(*i*) If we are looking for a definite decrease, *i. e.,* if H_1 is given by $\theta < \theta_0$ we use a one-sided left tail test.

(*ii*) If we are looking for a definite increase, *i. e.,* if H_1 is given by $\theta > \theta_0$ we use a one-sided right tail test.

(*iii*) If we are looking for any change, *i. e.,* if H_1 is given by $\theta \neq \theta_0$ we use a two-sided test.

The preceding discussion describing the nature of the one-tailed and two-tailed tests is summarized in the following table:

Table : Describing the nature of test.

Null hypothesis	Alternative hypothesis	Type of test	Location of rejection region
$\theta \geq \theta_0$	$\theta < \theta_0$	Left-tailed	The rejection region is located in the left tail of the sampling distribution under H_0.
$\theta \leq \theta_0$	$\theta > \theta_0$	Right-tailed	The critical region is located in the right tail of the sampling distribution under H_0.
$\theta = \theta_0$	$\theta \neq \theta_0$	Two-tailed	The rejection region is located equally in both tails of the sampling distribution under H_0.

Remarks **:** The names of the three types of a test are associated with alternative hypothesis H_1.

The choice of a one-sided left tail test, one-sided right tail test or a two-sided test depends upon the type of alternative hypothesis.

13.1.19 Errors of Inference. Sample evidence is used to test the null hypothesis H_0. If the sample evidence convinces us that H_0 has only a small chance of being true (a large chance of being false), we say H_0 is unreasonable and reject it. If the sample evidence does not convince us that H_0 is unreasonable, we do not reject H_0. Therefore, there are two alternative conclusions.

(*i*) Reject the null hypothesis on the basis of sample evidence.

(*ii*) Accept the null hypothesis on the basis of sample evidence.

There are two possible states of nature of the null hypothesis.

(*i*) The null hypothesis is true.

(*ii*) The null hypothesis is false.

Thus, in hypothesis testing one and only one of the following four possible outcomes will occur.

(*i*) If the null hypothesis is true, we may reject it leading to a wrong decision.

(*ii*) If the null hypothesis is true, we may accept or not reject it leading to a correct decision.

(*iii*) If the null hypothesis is false, we may accept it leading to a wrong decision.

(*iv*) If the null hypothesis is false, we may reject it leading to a correct decision.

Type-I Error. A Type-I error is made by rejecting H_0 if H_0 is actually true.

Type-II Error. A Type-II error is made by accepting H_0 if H_1 is actually true.

These four possible outcomes can be displayed in four cells of a table as follows:

Table : Hypothesis and conclusion reached from sample

Sample indicates conclusion	State of nature of hypotheses	
	H_0 is true	H_1 is true
Reject H_0	Type-I error	Correct decision
Do not reject (or accept) H_0	Correct decision	Type-II error

The probabilities of the four possible outcomes are commonly designated as

$$\alpha = P(\text{Rejecting } H_0 | H_0 \text{ is true}) = P(\text{Type-I error})$$

$$1 - \alpha = P(\text{Accepting } H_0 | H_0 \text{ is true})$$

$$\beta = P(\text{Accepting } H_0 | H_1 \text{ is true}) = P(\text{Type-II error})$$

$$1 - \beta = P(\text{Rejecting } H_0 | H_1 \text{ is true})$$

We can represent these conditional probabilities as:

Table : Conditional probabilities

Sample indicates conclusion	State of nature of hypotheses	
	H_0 is true	H_1 is true
Reject H_0	α	$1-\beta$
Do not reject (or accept) H_0	$1-\alpha$	β

13.1.20 Level of Significance. The *level of significance* of a test is the maximum probability with which we are willing to a risk of Type-I error. It is the probability of obtaining a value of the test statistic inside the critical region given that H_0 is true, *i. e.*, it is the probability of rejecting a true null hypothesis. It is denoted by α.

$$P(\text{Rejecting } H_0 | H_0 \text{ is true}) = P(\text{Type-I error}) = \alpha$$

The level of significance is also called as the *size of the critical region* or the *size of the test*. It is a small pre-assigned value, say, 0.05 or 0.01, which is generally specified before any samples are drawn, so that the results obtained will not influence our choice.

13.1.21 Level of Confidence. The *level of confidence* is the probability of accepting a true null hypothesis. It is denoted by $1 - \alpha$.

$$P(\text{Accepting } H_0 | H_0 \text{ is true}) = 1 - \alpha$$

Example 13.2 *The prosecuting attorney in a trial attempts to show that the defendant is guilty. The trail can be thought of a test of hypothesis, since a decision is to be made as to whether the defendant is guilty or innocent.*

(*a*) *State the null and alternative hypothesis of interest to the prosecuting attorney.*

(*b*) *Define the Type-I error and Type-II error for this situation.*

Solution. Since the prosecuting attorney wants to show that the defendant is guilty, this specifies the alternative hypothesis and the hypotheses are

H_0: Defendant is innocent $\qquad H_1$: Defendant is guilty

Type-I error would occur if the defendant were found guilty if, infect, the defendant is innocent.

Type-II error would occur if the defendant were found innocent if, infect, the defendant is guilty.

13.1.22 Statistical Decision Rule. A *statistical decision rule specifies*, for each possible sample outcome, whether the null hypothesis should be rejected or not. It is also called as a *decision function* or a *decision criterion*.

13.1.23 Conclusion. A test of hypothesis leads to one of two conclusions.

(*i*) If the observed value of the test statistic falls in the rejection region, the null hypothesis H_0 is rejected in favour of alternative hypothesis H_1.

(*ii*) If the observed value of the test statistic does not fall in the rejection region the null hypothesis is neither accepted nor rejected. It is, then, stated that there is insufficient evidence to make a decision.

13.1.24 Test of significance. The *test of significance* describes a process of testing a hypothesis to gain a general impression about the parameter. No decision is imminent or even implied. In this case, a level of significance is not present. Instead we investigate the values of α that would lead to rejection of H_0 as opposed to the α values leading to acceptance of H_0. The test of significance differs with hypothesis testing that refers to the act of actually testing a hypothesis at a selected level of significance to make a decision based on that conclusion.

13.1.25 Steps to Follow when Testing a Hypothesis. Since we will be conducting many tests of hypothesis, it is useful to follow a set procedure. In the remainder of this text, we will always follow the same format. The steps we will follow are given below:

***Before any sample observations are considered*:**

1. Identify the population of interest and state the conditions required for the validity of the test procedure being used.
2. Formulate and state the null hypothesis H_0 and the alternative hypothesis H_1.
3. Decide and specify the level of significance, α.
4. Select the appropriate test statistic and its sampling distribution if H_0 is assumed to be true.
5. Give the critical value (or values) of test statistic for desired value of α.
6. Establish the rejection (critical) region.
7. State the decision rule: Reject H_0 if the value of the test statistic from the observed sample falls in the rejection region, otherwise do not reject (or accept) H_0.

***Now consider the sample values*:**

8. Calculate the value of the test statistic from the observe sample.
9. In the light of your decision rule, draw the conclusion as to whether to reject or accept H_0 and then state the decision in managerial terms.

Exercise 13.1

1. (*a*) Define the following concepts in your own words as fully as you can:

   (*i*) Hypotheses testing (*ii*) Statistical hypothesis
   (*iii*) Null hypothesis (*iv*) Test statistic
   (*v*) Level of significance (*vi*) Critical region

   (*b*) Explain with examples the difference between:

   (*i*) Estimation and hypothesis testing
   (*ii*) Null hypothesis and alternative hypothesis
   (*iii*) Simple hypothesis and composite hypothesis
   (*iv*) Acceptance region and rejection region
   (*v*) Type-I error and Type-II error
   (*vi*) One-tailed test and two-tailed test

   (*c*) What is the difference between a one-sided and a two-sided test? When should each be used?

2. (*a*) Explain what is meant by:

   (*i*) Statistical hypothesis (*ii*) Test-statistic
   (*iii*) Significance level (*iv*) Test of significance

   (*b*) Distinguish between the following concepts:

   (*i*) Statistical estimation and hypothesis testing
   (*ii*) Rejection and non-rejection regions
   (*iii*) A test at α level of significance and $1 - \alpha$ confidence level

3. (*a*) Explain how the null hypothesis and the alternative hypothesis are formulated.

   (*b*) State the null and alternative hypotheses to be used in testing the following claims:

   (*i*) The mean rainfall at Lake Placid during the month of June is 2.8 inches.
   (*ii*) No more than 20% of the faculty at Highland University contributed to the annual giving fund.
   (*iii*) The proportion of voters favouring Senator Foghorn in up coming election is 0.58.
   (*iv*) On the average children attend school within 3.8 miles of their homes in suburban San Francisco.

   { (*i*) $H_0: \mu = 2.8$ inches against $H_1: \mu \neq 2.8$ inches; (*ii*) $H_0: \pi \leq 0.20$ against $H_1: \pi > 0.20$; (*iii*) $H_0: \pi = 0.58$ against $H_1: \pi \neq 0.58$; (*iv*) $H_0: \mu \leq 3.8$ miles against $H_1: \mu > 3.8$ miles.}

4. (*a*) Indicate Type-I and Type-II errors in the following statements:

   (*ii*) A judge can acquit a guilty person.
   (*iii*) A deserving player may not be selected in the team.
   (*iv*) A bad student may be passed by the examiner.

   { (*i*) Type-I, (*ii*) Type-II, (*iii*) Type-I, (*iv*) Type-II.}

   (*b*) For each of the following situations, indicate the Type-I and Type-II errors and the correct decisions.

   (*i*) H_0: New system is no better than the old one.

   Adopt new system when new one is better.
   Retain old system when new one is better.
   Retain old system when new one is not better.
   Adopt new system when new one is not better

   (*ii*) H_0: New product is satisfactory.

   Market new product when unsatisfactory.
   Do not market new product when unsatisfactory.
   Do not market new product when satisfactory.
   Market new product when satisfactory.

   { (*i*) Correct, Type-II, Correct, Type-I; (*ii*) Type-II, Correct, Type-I, Correct }

   (*c*) Suppose that a psychological testing service is asked to check whether an executive is emotionally fit to assume the presidency of a large company. What type of error is committed if the hypothesis that he is fit for the job is erroneously accepted? What type of error is committed if the hypothesis that he is fit for the job is erroneously rejected ? (Type-II error; Type-I error)

5. (*a*) The director of an advertising agency is concerned with the effectiveness of a certain kind of television commercial.

   (*i*) What hypothesis is he testing, if he is committing a Type-I error when he says erroneously that the commercial is effective.
   (*ii*) What hypothesis is he testing, if he is committing a Type-II error when he says erroneously that the commercial is effective.

   { (*i*) Commercial is not effective; (*ii*) Commercial is effective }

   (*b*) Outline the fundamental procedure followed in testing a null hypothesis.

13.2 TEST OF HYPOTHESIS ABOUT A POPULATION MEAN, μ

13.2.1 Forms of Hypothesis. Let μ_0 be the hypothesized value of a population mean. The three possible null hypotheses about a population mean, and their corresponding alternative hypotheses, are:

1. $H_0: \mu \geq \mu_0$ against $H_1: \mu < \mu_0$
2. $H_0: \mu \leq \mu_0$ against $H_1: \mu > \mu_0$
3. $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$

In each case, the test will be made by obtaining a simple random sample of size n and computing the sample mean $\bar{X}$. Then $\bar{X}$ will be used in computing a test statistic. Depending upon the calculated value of the test statistic, H_0 will be accepted or rejected.

13.2.2 Normal Population, σ^2 known. We know that for a normal population having a mean μ_0 and a known variance σ^2 the sampling distribution of $\bar{X}$ is normal with mean μ_0 and a standard error of $\sigma_{\bar{X}} = \sigma/\sqrt{n}$. Then the statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

has the standard normal distribution. Consequently, the statistic Z is the test statistic for testing a hypothesis about the mean of a normal population whose variance σ^2 is known.

13.2.3 Normal Population, σ^2 unknown. If the variance of the population σ^2 is unknown, then we estimate σ^2 by the sample variance $\hat{S}^2$ and we estimate the standard error of $\bar{X}$ by $S_{\bar{X}}$, where $S_{\bar{X}} = \hat{\sigma}_{\bar{X}} = \hat{S}/\sqrt{n}$. If the population is normal, the statistic

$$T = \frac{\bar{X} - \mu_0}{S_{\bar{X}}} = \frac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$$

has a Student's t-distribution with $\nu = n - 1$ degrees of freedom. Consequently the statistic T is the test statistic for testing a hypotheses about the mean of a normal population whose variance σ^2 is unknown.

13.2.4 Any Population, σ^2 known/unknown. Finally recall the Central Limit Theorem which states that the sampling distribution of $\bar{X}$ is approximately normal even for non-normal populations if the sample size is sufficiently large (say, $n > 30$). Then we will use the test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \cong \frac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$$

to test a hypothesis about the mean of any population (normal or not).

The summary of the procedure of testing a hypothesis about the mean of a population whose variance σ^2 is known/unknown, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing mean of a population, σ^2 known/unknown:
Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \mu \geq \mu_0$ $H_1: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_1: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$

Set level of significance: α

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \cong \frac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$$

$$\bar{X} = \frac{\sum X_i}{n}, \qquad \hat{S}^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$$

Reject H_0 if $Z < z_\alpha$	Reject H_0 if $Z > z_{1-\alpha}$	Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

Calculate the observed value of test statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \cong \frac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}}$$

$$\bar{x} = \frac{\sum x_i}{n}, \qquad \hat{s}^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

We reject H_0 if $z < z_\alpha$ otherwise we do not reject H_0	We reject H_0 if $z > z_{1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ otherwise we do not reject H_0

The summary of the procedure of testing a hypothesis about the mean of a normal population whose variance σ^2 is unknown, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing mean of a normal population, σ^2 unknown: Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \mu \geq \mu_0$ $H_1: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_1: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$

Set level of significance: α

$$T = \frac{\overline{X} - \mu_0}{\hat{S}/\sqrt{n}}$$

$$\overline{X} = \frac{\Sigma X_i}{n}, \quad \hat{S}^2 = \frac{\Sigma(X_i - \overline{X})^2}{n-1}$$

Reject H_0 if $T < t_{v;\alpha}$ $v = n - 1$	Reject H_0 if $T > t_{v;1-\alpha}$	Reject H_0 if $T < t_{v;\alpha/2}$ or $T > t_{v;1-\alpha/2}$

Calculate the observed value of test statistic

$$t = \frac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}}$$

$$\bar{x} = \frac{\Sigma x_i}{n}, \quad \hat{s}^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$$

We reject H_0 if $t < t_{v;\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t > t_{v;1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t < t_{v;\alpha/2}$ or $t > t_{v;1-\alpha/2}$ otherwise we do not reject H_0

Choice of Test Statistic for Testing Population Mean. The appropriate test statistic for testing a mean is either the Z or the T statistic, and depends upon the sample size and whether the

population distribution and population variance are known. Thus in testing the hypothesis about the population mean we will use:

(*i*) The test statistic Z when the population is normal and variance is known, or the sample size is sufficiently large,

(*ii*) The test statistic T when the population is normal, whose variance is unknown and sample size is small ($n \leq 30$).

This can be summarized as follows.

Population Distribution	Population Variance σ^2	Sample Size n	Appropriate Test Statistic
Normal	Known	Any	Z
Normal	Unknown	Any	T (If $n > 30$, can also use Z as an approximation)
Any	Known or unknown	$n > 30$	Z

where, when σ^2 is unknown, its estimate is the sample variance $\hat{S}^2$.

Example 13.3 *A Bureau of Research statistician is designing an experiment to test achievement scores of first year students in Government College Gujranwala on a standardized test, scored one through one hundred. He is willing to assume that the distribution of scores is normal. He also believes that the true achievement of all first years students in Government College, Gujranwala is greater than 58. If he has a sample of 36 such scores what hypothesis should he test.*

Solution. Since he believes $\mu > 58$, he should test $H_0: \mu \leq 58$ against $H_1: \mu > 58$. He hopes to reject H_0. A rejection of H_0 is a positive action due to the overwhelming evidence that μ is not less than or equal to 58. An acceptance of H_0 is a negative action because it implies that there is not sufficient evidence to reject H_0.

Example 13.4 *If the null hypothesis is $H_0: \mu \leq 50$, when would a Type-II error be made.*

Solution. When infact $\mu > 50$ and H_0 is accepted.

Example 13.5 *If the null hypothesis is $H_0: \mu \geq 100$, when would a Type-II error be made.*

Solution. When infact $\mu < 100$ and H_0 is accepted.

Example 13.6 *A company claims that the average amount of coffee it supplies in jars is 6.0 oz with a standard deviation of 0.2 oz. A random sample of 100 jars is selected and average is found to be 5.9. Is the company cheating the customers? Use 5% level of significance.*

Solution. We have $n = 100$, $\bar{x} = 5.9$, $\sigma = 0.2$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the average amount μ is less than 6.0 ounces. Consequently, we will test the null hypothesis that $\mu \geq 6.0$ against the alternative hypothesis that $\mu < 6.0$.

The elements of the one-sided left tail test of hypothesis are: We State our Hypothesis as.

Null hypothesis $H_0: \mu \geq 6.0$

Alternative hypothesis $H_1: \mu < 6.0$

Level of significance: $\alpha = 0.05$

Test statistic: $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ follows a standard normal distribution under H_0

Critical values: $z_\alpha = z_{0.05} = -1.645$ { From Table 10 (*b*) }

Critical region: $Z < -1.645$

Decision rule: Reject H_0 if $Z < -1.645$, otherwise do not reject H_0

Observed value: $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{5.9 - 6.0}{0.2/\sqrt{100}} = -5.0$

Conclusion: Since $z = -5 < -1.645$, we reject H_0 and conclude that the average amount of coffee is less than 6.0 ounces.

Example 13.7 *A company makes parachutes. The company has been buying snap links from a manufacturing firm. The company is concerned that the quality of snap links they receive from the firm might not be up to specifications. Specifically, the company wants to be convinced that snap links will withstand a mean breaking strength of more than 5000 pounds. Perform a test of hypothesis at the 0.005 significance level if the mean breaking strength for a random sample of 50 snap links is 5100 pounds. The population standard deviation is 221 pounds. What is implied by the test result.*

Solution. We have $n = 50$, $\bar{x} = 5100$, $\sigma = 221$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean breaking strength μ is more than 5000 pounds. Consequently, we will test the null hypothesis that $\mu \le 5000$ against the alternative hypothesis that $\mu > 5000$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu \le 5000$

Alternative hypothesis $H_1: \mu > 5000$

Level of significance: $\alpha = 0.005 \Rightarrow 1 - \alpha = 1 - 0.005 = 0.995$

Test statistic: $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ follows a standard normal distribution under H_0.

Critical value: $z_{1-\alpha} = z_{0.995} = 2.576$ { From Table 10 (*b*) }

Critical region: $Z > 2.576$

Decision rule: Reject H_0 if $Z > 2.576$, otherwise do not reject H_0.

Observed value: $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{5100 - 5000}{221/\sqrt{50}} = 3.20$

Conclusion: Since $z = 3.20 > 2.576$, we reject H_0 and conclude that the mean breaking strength is more than 5000 pounds. Links from the firm are up to specifications. Buy them.

Example 13.8 *The mean lifetime of electric light bulbs produced by a company has in the past been 1120 hours. A random sample of 36 electric bulbs recently chosen from a supply of newly manufactured bulbs showed a mean lifetime of 1087 hours with a standard deviation of 120 hours. Test the hypothesis that the mean lifetime of light bulbs has not changed using a level of significance of 0.05.*

Solution. We have $n = 36$, $\bar{x} = 1087$, $\hat{s} = 120$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean lifetime μ has changed from 1120 hours. Consequently, we will test the null hypothesis that $\mu = 1120$ against the alternative hypothesis that $\mu \ne 1120$.

The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \mu = 1120$

Alternative hypothesis $H_1: \mu \ne 1120$

Level of significance: $\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow 1 - \alpha/2 = 0.975$

Test statistic: $Z = \frac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows an approximate standard normal distribution under H_0.

Critical values: $z_{\alpha/2} = z_{0.025} = -1.960$, $z_{1-\alpha/2} = z_{0.975} = 1.960$ { From Table 10 (*b*) }

Critical region: $Z < -1.960$ or $Z > 1.960$

Decision rule: Reject H_0 if $Z < -1.960$ or $Z > 1.960$, otherwise do not reject H_0.

Observed value: $z = \frac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}} = \frac{1087 - 1120}{120/\sqrt{36}} = -1.65$

Conclusion: Since $-1.960 < z = -1.65 < 1.960$, we do not reject H_0 and conclude that the mean lifetime is 1120 hours.

Example 13.9 *We wish to test the hypothesis that the mean weight of a population of people is 140 lb. Using $\sigma = 15$ lb. $\alpha = 0.05$ and a sample of 36 people, find. the values of $\bar{X}$ which would lead to rejection of the hypothesis.*

Solution. We have $n = 36$, $\sigma = 15$.

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean weight μ is not equal to 140 pounds. Consequently, we will test the null hypothesis that $\mu = 140$ against the alternative hypothesis that $\mu \ne 140$.

The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \mu = 140$

Alternative hypothesis $H_1: \mu \ne 140$

Level of significance: $\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow 1 - \alpha/2 = 0.975$

Test statistic: $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ follows a standard normal distribution under H_0

Critical values: $z_{\alpha/2} = z_{0.025} = -1.960$, $z_{1-\alpha/2} = z_{0.975} = 1.960$ { From Table 10 (*b*) }

Critical region: $Z < -1.960$ or $Z > 1.960$

Decision rule: Reject H_0 if $Z < -1.960$ or $Z > 1.960$

$$\frac{\overline{X} - 140}{15/\sqrt{36}} < -1.960 \text{ or } \frac{\overline{X} - 140}{15/\sqrt{36}} > 1.960$$

$$\overline{X} < 135.1 \quad \text{or} \quad \overline{X} > 144.9$$

Hence, the hypothesis $H_0: \mu = 140$ will be rejected if either $\overline{X} < 135.1$ or $\overline{X} > 144.9$.

Example 13.10 *An ultrasonic equipment manufacturer uses a component in one of its machines that must withstand considerable stress from vibrations while the machine is operating. An all-metal component has been available for some years from a supplier. Extensive historical experience has shown this component has a mean service life of 1100 hours. The research division of the supplier has just developed a modified component constructed from plastic and metal, bonded in a special way. The manufacturer wishes to know whether or not the mean service life of the modified component (μ) exceeds the mean service life of 1100 hours for the original component. Test the hypothesis at 1% level of significance if a sample of $n = 36$ components yielded $\bar{x} = 1121$ and $\hat{s} = 222$ hours.*

Solution. We have $n = 36$, $\bar{x} = 1121$, $\hat{s} = 222$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean service life μ exceeds 1100 hours. Consequently, we will test the null hypothesis that $\mu \le 1100$ against the alternative hypothesis that $\mu > 1100$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu \le 1100$

Alternative hypothesis $H_1: \mu > 1100$

Level of significance: $\alpha = 0.01 \Rightarrow 1 - \alpha = 1 - 0.01 = 0.99$

Test statistic: $Z = \frac{\overline{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows a approximate standard normal distribution under H_0.

Critical value: $z_{1-\alpha} = z_{0.99} = 2.326$ { From Table 10 (b) }

Critical region: $Z > 2.326$

Decision rule: Reject H_0 if $Z > 2.326$, otherwise do not reject H_0.

Observed value: $z = \frac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}} = \frac{1121 - 1100}{222/\sqrt{36}} = 0.568$

Conclusion: Since $z = 0.568 < 2.326$, so we do not reject H_0 and conclude that the mean service life does not exceeds 1100 hours.

Example 13.11 *A lumber company is interested in seeing if the number of board feet per tree has decreased since moving to a new location of timber. In the past, the company has an average of 93 board feet per tree. The company believes that the production has decreased since changing locations, a random sample of 25 trees yields $\bar{x} = 89$ with $\hat{s} = 20$. Assuming the normality of the data, test the hypothesis at a 10% level of significance.*

Solution. We have $n = 25$, $\bar{x} = 89$, $\hat{s} = 20$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean production μ has decreased from 93 board feet. Consequently, we will test the null hypothesis that $\mu \ge 93$ against the alternative hypothesis that $\mu < 93$.

The elements of the one-sided left tail test of hypothesis are:

Null hypothesis $H_0: \mu \ge 93$

Alternative hypothesis $H_1: \mu < 93$

Level of significance: $\alpha = 0.10$

Test statistic: $T = \frac{\overline{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $v = n - 1 = 25 - 1 = 24$

Critical value: $t_{v;\alpha} = t_{24;0.10} = -1.318$ (From Table 12)

Critical region: $T < -1.318$

Decision rule: Reject H_0 if $T < -1.318$, otherwise do not reject H_0.

Observed value: $t = \frac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}} = \frac{89 - 93}{20/\sqrt{25}} = -1.000$

Conclusion: Since $t = -1.000 > -1.318$, we do not reject H_0 and conclude that the mean production has not decreased from 93 board feet. The tree company's claim has not been established.

Example 13.12 *Expensive test borings were made in an oil shale area to determine if the mean yield of oil per ton of shale rock is greater than 4.5 barrels. Five borings, made at randomly selected points in the area, indicated the following number of barrels per ton 4.8, 5.4, 3.9, 4.9, 5.5. Suppose barrels per ton are normally distributed. Perform a test of hypothesis at the 5% level of significance.*

Solution. The mean and standard deviation of the sample are

x_i	4.8	5.4	3.9	4.9	5.5	$\sum x_i = 24.5$
x_i^2	23.04	29.16	15.21	24.01	30.25	$\sum x_i^2 = 121.67$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{24.5}{5} = 4.9$$

$$\hat{s} = \sqrt{\frac{\sum x_i^2 - n\bar{x}^2}{n - 1}} = \sqrt{\frac{121.67 - 5(4.9)^2}{5 - 1}} = 0.64$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean yield μ is greater than 4.5 barrels. Consequently, we will test the null hypothesis that $\mu \le 4.5$ against the alternative hypothesis that $\mu > 4.5$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu \leq 4.5$

Alternative hypothesis $H_1: \mu > 4.5$

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $T = \dfrac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n - 1 = 5 - 1 = 4$

Critical value: $t_{\nu;1-\alpha} = t_{4;0.95} = 2.132$ (From Table 12)

Critical region: $T > 2.132$

Decision rule: Reject H_0 if $T > 2.132$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}} = \dfrac{4.9 - 4.5}{0.64/\sqrt{5}} = 1.40$

Conclusion: Since $t = 1.40 < 2.132$, we do not reject H_0 and conclude that the mean yield is not greater than 4.5 barrels..

Example 13.13 *A cattle rancher has changed the type of feed he uses to fatten his cattle for sale. The feed company claims that the new feed will increase the mean weight gain in his cattle by more than 100 pounds per steer. Assuming the weight gain of cattle is normally distributed, test the hypothesis of the feed company at $\alpha = 0.05$. Previously, the mean weight gain per steer has been 800 pounds. A random sample of 30 yields a mean weight gain of $\bar{x} = 935$ pounds with a standard deviation of 85 pounds.*

Solution. We have $n = 30$, $\bar{x} = 935$, $\hat{s} = 85$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean weight μ is more than 900 (*i. e.*, 800 + 100) pounds. Consequently, we will test the null hypothesis that $\mu \leq 900$ against the alternative hypothesis that $\mu > 900$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu \leq 900$

Alternative hypothesis $H_1: \mu > 900$

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $T = \dfrac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n - 1 = 30 - 1 = 29$

Critical value: $t_{\nu;1-\alpha} = t_{29;0.95} = 1.699$ (From Table 12)

Critical region: $T > 1.699$

Decision rule: Reject H_0 if $T > 1.699$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\bar{x} - \mu_0}{\hat{s}/\sqrt{n}} = \dfrac{935 - 900}{85/\sqrt{30}} = 2.26$

Conclusion: Since $t = 2.26 > 1.699$, so we reject H_0 and conclude that the mean weight is more than 900 (*i. e.*, 800 + 100) pounds.. The feed company's claim has been established.

Example 13.14 *Workers at a production facility are required to assemble a certain part in 2.3 minutes in order to meet production criteria. The assembly rate per part is assumed to be normally distributed. Six workers are selected at random and timed in assembling a part. The assembly times (in minutes) for the six workers as follows.*

2.0 2.4 1.7 1.9 2.8 1.8

The manager wants to determine if the mean for all workers differs from 2.3. Perform a test of hypothesis at the 5% level of significance.

Solution. The mean and standard deviation of the sample are

x_i	2.0	2.4	1.7	1.9	2.8	1.8	$\sum x_i = 12.6$
$x_i - \bar{x}$	– 0.1	0.3	– 0.4	– 0.2	0.7	– 0.3	
$(x_i - \bar{x})^2$	0.01	0.09	0.16	0.04	0.49	0.09	$\sum (x_i - \bar{x})^2 = 0.88$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{12.6}{6} = 2.1$$

$$\hat{s} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{0.88}{6-1}} = 0.42$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean assembly time μ differs from 2.3 minutes. Consequently, we will test the null hypothesis that $\mu = 2.3$ against the alternative hypothesis that $\mu \neq 2.3$.

The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \mu = 2.3$

Alternative hypothesis $H_1: \mu \neq 2.3$

Level of significance: $\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow 1 - \alpha/2 = 0.975$

Test statistic: $T = \dfrac{\bar{X} - \mu_0}{\hat{S}/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n - 1 = 6 - 1 = 5$

Critical values: $t_{\nu;\alpha/2} = t_{5;0.025} = -2.571$,

$t_{\nu;1-\alpha/2} = t_{5;0.975} = 2.571$ (From Table 12)

Critical region: $T < -2.571$ or $T > 2.571$

Decision rule: Reject H_0 if $T < -2.571$ or $T > 2.571$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{2.1 - 2.3}{0.42/\sqrt{6}} = -1.166$

Conclusion: Since $-2.571 < t = -1.166 < 2.571$, we do not reject H_0 and conclude that the average assembly time is 2.3 minutes.

Exercise 13.2

1. (a) Why is the z-test usually inappropriate as a test-statistic when the sample size is small?

(b) Define "Student's t-statistic". What are its assumptions? Explain briefly its use and importance in statistics.

2. For each of the following, a random sample of size n is taken from a normal distribution with mean μ and variance σ^2. The sample mean is $\bar{x}$. Test the hypothesis stated, at the level of significance indicated.

Sample	n	$\bar{x}$	σ	Hypotheses H_0	Hypotheses H_1	Level of significance
(i)	30	15.2	3	$\mu = 15.8$	$\mu \neq 15.8$	5%
(ii)	10	27	1.2	$\mu \leq 26.3$	$\mu > 26.3$	5%
(iii)	49	125	4.2	$\mu \leq 123.5$	$\mu > 123.5$	1%
(iv)	100	4.35	0.18	$\mu \geq 4.40$	$\mu < 4.40$	2%

Sample	n	$\bar{x}$	$\sum(x_i - \bar{x})^2$	Hypotheses H_0	Hypotheses H_1	Level of significance
(v)	65	100	842.4	$\mu = 99.2$	$\mu \neq 99.2$	5%
(vi)	65	100	842.4	$\mu \leq 99.2$	$\mu > 99.2$	5%
(vii)	80	85.3	2508.8	$\mu \geq 86.2$	$\mu < 86.2$	10%
(viii)	100	6.85	36	$\mu = 7.0$	$\mu \neq 7.0$	1%

[(i) Since $z_{0.025} = -1.96 < z = -1.095 < 1.96 = z_{0.975}$, we do not reject $H_0: \mu = 15.8$ against $H_1: \mu \neq 15.8$.

(ii) Since $z = 1.845 > 1.645 = z_{0.95}$, we reject $H_0: \mu \leq 26.3$ in favour of $H_1: \mu > 26.3$.

(iii) Since $z = 2.5 > 2.326 = z_{0.99}$, we reject $H_0: \mu \leq 123.5$ in favour of $H_1: \mu > 123.5$.

(iv) Since $z = -2.778 < -2.054$, we reject $H_0: \mu \geq 4.40$ in favour of $H_1: \mu < 4.40$.

(v) Since $z_{0.025} = -1.96 < z = 1.778 < 1.96 = z_{0.975}$, we do not reject $H_0: \mu = 99.2$ against $H_1: \mu \neq 99.2$.

(vi) Since $z = 1.792 > 1.645 = z_{0.95}$, we reject $H_0: \mu \leq 99.2$ in favour of $H_1: \mu > 99.2$.

(vii) Since $z = -1.428 < -1.282 = z_{0.10}$, we reject $H_0: \mu \geq 86.2$ in favour of $H_1: \mu < 86.2$.

(viii) Since $z_{0.005} = -2.576 < z = -2.5 < 2.576 = z_{0.995}$, we do not reject $H_0: \mu = 7.0$ against $H_1: \mu \neq 7.0$.]

3. (a) A random sample of size 36 is taken from a normal population with known variance $\sigma^2 = 25$. If the mean of the sample is $\bar{x} = 42.6$, test the null hypothesis $\mu \geq 45$ against the alternative hypothesis $\mu < 45$ with $\alpha = 0.05$ (α is the probability of committing Type-I error).
(Since $z = -2.88 < -1.645 = z_{0.05}$, we reject $H_0: \mu \geq 45$ in favour of $H_1: \mu < 45$)

(b) Ten dry cells were taken from store and voltage tests gave the following results: 1.52, 1.53, 1.49, 1.48, 1.47, 1.49, 1.51, 1.50, 1.47, 1.48 volts. The mean voltage of the cells when stored was 1.51V. Assuming the population standard deviation to remain unchanged at 0.02 V, is there reason to believe that the cells have deteriorated.
(Since $z = -2.53 < -1.645 = z_{0.05}$, we reject $H_0: \mu \geq 1.51$ in favour of $H_1: \mu < 1.51$.)

4. (a) A sample of 400 male students is found to have a mean height of 67.47 inches. Can it be regarded as a simple random sample from a large population with mean height 67.39 with standard deviation of 1.3 inches?
(Since $z_{0.025} = -1.96 < z = 1.23 < 1.96 = z_{0.975}$, we do not reject $H_0: \mu = 67.39$ against $H_1: \mu \neq 67.39$ at $\alpha = 0.05$)

(b) The mean lifetime of electric light bulbs produced by a company has in the past been 1120 hours with a standard deviation of 125 hours. A sample of 8 electric bulbs recently chosen from a supply of newly manufactured bulbs showed a mean lifetime of 1070 hours. Test the hypothesis that mean lifetime of the bulbs has not changed using a level of significance of 0.05.
(Since $z_{0.025} = -1.96 < z = -1.13 < 1.96 = z_{0.975}$, we do not reject $H_0: \mu = 1120$ against $H_1: \mu \neq 1120$)

5. (a) Suppose that the mean μ of a random variable X is unknown but the variance of X is known to be 144. Should we reject the null hypothesis $H_0: \mu = 15$ in favour of an alternative hypothesis $H_1: \mu \neq 15$ at a level of significance of $\alpha = 0.05$, if a random sample of 64 observations yields a mean $\bar{x} = 12$? What are the 95% confidence limits for μ.
(Since $z = -2 < -1.960 = z_{0.025}$, we reject $H_0: \mu = 15$ in favour of $H_1: \mu \neq 15$; $9.06 < \mu < 14.94$)

(b) In a syrup filling factory the mean weight per bottle is claimed to be $\mu = 16$. A random sample of 100 bottles is taken and a test statistic used is $\overline{X}$, the mean weight of the sample. For a 0.05 level of significance, find the critical values for the test statistic and

formulate the decision rule. Assume the weights to be normally distributed with $\sigma^2 = 2.56$.
(Reject H_0 if $\bar{X} < 15.68$ or $\bar{X} > 16.32$)

6. (a) A random sample of 36 drinks from a soft-drink machine has an average content 7.6 ounces with a standard deviation of 0.48 ounces. Test the hypothesis $\mu \le 7.5$ ounces against the alternative hypothesis $\mu > 7.5$ at the 0.05 level of significance.
(Since $z = 1.25 < 1.645 = z_{0.95}$, we do not reject $H_0: \mu \le 7.5$ against $H_1: \mu > 7.5$)

(b) It is claimed that an automobile is driven on the average at most 20000 kilometres per year. To test this claim, a random sample of 100 automobile owners are asked to keep a record of the kilometres they travel. Would you agree with claim if the random sample showed an average of 21500 kilometres and a standard deviation of 3900 kilometres? Use a 0.01 level of significance.
(Since $z = 3.846 > 2.326 = z_{0.990}$, we reject $H_0: \mu \le 20000$ in favour of $H_1: \mu > 20000$)

7. (a) A random sample of 100 recorded deaths in the United States during the past year showed and average lifespan of 71.8 years with a standard deviation of 8.9 years. Does this seem to indicate that the average lifespan today is greater than 70 years? Use a 0.05 level of significance.
(Since $z = 2.02 > 1.645 = z_{0.95}$, we reject $H_0: \mu \le 70$ in favour of $H_1: \mu > 70$)

(b) It is claimed that an automobile is driven on the average no more than 12000 miles per year. To test this claim, a random sample of 100 automobile owners are asked to keep a record of the miles they travel. Would you agree with the claim if the random sample showed an average of 12500 miles and a standard deviation of 2400 miles?
(Since $z = 2.083 > 1.645$, we reject $H_0: \mu \le 12000$ in favour of $H_1: \mu > 12000$ at $\alpha = 0.05$)

8. (a) Given the following information. What is your conclusion in testing each of the indicated null hypothesis? Assume the populations are normal.

Sample	Sample size	Sample mean	Estimate of variance from sample	Hypotheses H_0	Hypotheses H_1	level of significance
(i)	16	11	81	$\mu \le 10$	$\mu > 10$	0.01
(ii)	25	8	64	$\mu \ge 10$	$\mu < 10$	0.01
(iii)	25	9	49	$\mu = 10$	$\mu \ne 10$	0.02
Sample	n	$\bar{x}$	$\sum(x_i - \bar{x})^2$	Hypotheses H_0	Hypotheses H_1	Level of significance
(iv)	12	24.9	12.3	$\mu \le 24.1$	$\mu > 24.1$	2.5%
(v)	17	35.6	1471.8	$\mu = 40$	$\mu \ne 40$	5%
(vi)	6	1505.8	50.8	$\mu \le 1503$	$\mu > 1503$	5%
(vii)	10	129.8	97.6	$\mu \ge 133.0$	$\mu < 133.0$	1%

{ (i) Since $t = 0.44 < 2.602 = t_{15;\,0.99}$, we do not reject H_0.

(ii) Since $t = -1.25 > -2.492 = t_{24;\,0.01}$, we do not reject H_0.

(iii) Since $t_{24;\,0.01} = -2.492 < t = -0.714 < 2.492 = t_{24;\,0.99}$, we do not reject H_0.

(iv) Since $t = 2.622 > 2.201 = t_{11;\,0.975}$, we reject $H_0: \mu \le 24.1$ in favour of $H_1: \mu > 24.1$.

(v) Since $t_{16;\,0.25} = -2.120 < t = -1.892 < 2.120 = t_{16;\,0.975}$ we do not reject $H_0: \mu = 40$ against $H_1: \mu \ne 40$.

(vi) Since $t = 2.152 > 2.015 = t_{5;\,0.95}$, we reject $H_0: \mu \le 1503$ in favour of $H_1: \mu > 1503$.

(vii) Since $t = -3.073 < -2.821 = t_{9;\,0.01}$, we reject $H_0: \mu \ge 133.0$ in favour of $H_1: \mu < 133.0$ }.

(b) A random sample of size n is drawn from normal population with mean 5 and variance σ^2.

(i) If $n = 25$, $\bar{x} = 3$ and $\hat{s} = 2$, what is t?

(ii) If $n = 9$, $\bar{x} = 2$ and $t = -2$, what is $\hat{s}$?

(iii) If $n = 25$, $\hat{s} = 10$ and $t = 2$, what is $\bar{x}$

(iv) If $\hat{s} = 15$, $\bar{x} = 14$ and $t = 3$, what is n?
($t = -5$, $\hat{s} = 4.5$, $\bar{x} = 9$, $n = 25$)

9. (a) Injection of a certain type of hormone into hens is said to increase the mean weight of eggs by 0.3 ounces. A sample of 30 eggs has an arithmetic mean 0.4 ounces above the pre-injection mean and a value of $\hat{s}$ equal to 0.20. Is this enough reason to accept the statement that the mean increase is more than 0.3 ounces?
(Since $t = 2.74 > 1.699 = t_{29;\,0.95}$, we reject $H_0: \mu \le 0.3$ in favour of $H_1: \mu > 0.3$ at $\alpha = 0.05$.)

(b) A random sample of 25 hens from a normal population showed that the average laying is 272 eggs per year with a variance of 625 eggs. The company claimed that the average laying is at least 285 eggs per year. Test the claim of the company at $\alpha = 0.05$.
(Since $t = -2.6 < -1.711 = t_{24;\,0.05}$, we reject $H_0: \mu \ge 285$ in favour of $H_1: \mu < 285$).

10. (a) A producer of a certain make of flashlight dry cell batteries claims that its output has a mean life of 750 minutes. A random sample of 15 such batteries has been tested and a sample mean of 745 minutes and a sample standard deviation of 24 minutes have been

obtained. Verify that these results are consistent with the null hypothesis $\mu \geq 750$ against $\mu < 750$ at $\alpha = 0.01$.
(Since $t = -0.807 > -2.624 = t_{14;0.01}$, we do not reject $H_0: \mu \geq 750$ against $H_1: \mu < 750$.)

(*b*) Ten individuals are chosen at random from a normal population and the heights are found to be 63, 63, 66, 67, 67, 69, 70, 70, 71 and 71inches. In the light of these data, discuss the suggestion that the mean height in the population is 66 inches.
(Since $t_{9;0.025} = -2.262 < t = 1.78 < 2.262 = t_{9;0.975}$, we do not reject $H_0: \mu = 66$ against $H_1: \mu \neq 66$ at $\alpha = 0.05$.)

11. (*a*) Ten cartons are taken at random from an automatic filling machine. The mean net weight of the 10 cartons is 15.90 *oz* and the sum of squared deviations from this mean is 0.276 (oz)^2. Does the sample mean differ significantly from intended weight of 16 *oz*.?
(Since $t_{9;0.025} = -2.262 \leq t = -1.806 \leq 2.262 = t_{9;0.975}$, we do not reject $H_0: \mu = 16$ against $H_1: \mu \neq 16$ at $\alpha = 0.05$.)

(*b*) In the past a machine has produced washers having thickness of 0.050 inches. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is 0.053 inches and the standard deviation is 0.003 inches. Test the hypothesis that the machine is in proper working order using a level of significance of 0.05.
(Since $t = 3.16 > 2.262 = t_{9;0.975}$, we reject $H_0: \mu = 0.050$ in favour of $H_1: \mu \neq 0.050$.)

12. (*a*) A random sample of 16 values from a normal population showed a mean of 41.5 inches and a sum of squares of deviations from this mean equal to 135 (inches)^2. Show that the assumption of a mean of 43.5 inches for the population is not reasonable and that the 95% confidence limits for this mean are 39.9 and 43.1 inches.
(Since $t = -2.667 < -2.131 = t_{15;0.025}$, we reject $H_0: \mu = 43.5$ in favour of $H_1: \mu \neq 43.5$ at $\alpha = 0.05$.)

(*b*) A random sample of nine from the men of a large city gave a mean height of 68 inches, and the unbiased estimate of the population variance from sample, $\hat{s}^2$ was 4.5 (inches)^2. Are these data consistent with the assumption of a mean height of 68.5 inches for the men of the city.?
(Since $t_{8;0.025} = -2.306 < t = -0.708 < 2.306 = t_{8;0.975}$, so we do not reject $H_0: \mu = 68.5$ against $H_1: \mu \neq 68.5$ at $\alpha = 0.05$).

13.3 TEST OF HYPOTHESIS ABOUT A POPULATION PROPORTION, π

Tests concerning proportions are based on frequency or count data, which are outcomes of experiments such as the number of defective items in a production line, the number of errors made in typing a complex mathematical manuscript, and so forth. In this section we shall present the tests based on count data, where the test concerns the parameter π of the Bernoulli distribution for both small and large sample sizes. The statistic for testing hypothesis concerning proportions is the number of successes X or the proportion of successes P in n independently repeated Bernoulli trials. For small n, the tests require the use of binomial probabilities. For large n, the normal approximation to the binomial, using the Z statistic, is appropriate, the Z^2 statistic has χ^2-distribution.

13.3.1 Forms of Hypothesis. Let π_0 be the hypothesised value of the population proportion. The three possible null hypotheses and their corresponding alternative hypotheses, are:

1. $H_0: \pi \geq \pi_0$ against $H_1: \pi < \pi_0$
2. $H_0: \pi \leq \pi_0$ against $H_1: \pi > \pi_0$
3. $H_0: \pi = \pi_0$ against $H_1: \pi \neq \pi_0$

In each case, the test will be made by obtaining a simple random sample of size n and counting the number of successes X in the sample and computing the sample proportion P. Then P will be used in computing a test statistic. Depending upon the calculated value of the test statistic, H_0 will be accepted or rejected.

13.3.2 Test based on Normal Approximation. We know that for a Bernoulli distribution with proportion of success π_0, the sampling distribution of the number of successes X in a sample of size n is a binomial distribution with parameters n and π_0. The sampling distribution of the sample proportion $P = X/n$ is also a binomial distribution. The mean of the sampling distribution of P is π_0 and that $\pi_0(1-\pi_0)/n$ is the variance of the sampling distribution. Consequently, the test statistic is

$$Z = \frac{P - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$$

which is approximately standard normal for large sample because of the normal approximation to the binomial. Or multiplying the numerator and denominator by n, we obtain the test statistic.

$$Z = \frac{X - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}}$$

which is approximately standard normal when n is large and π_0 is not too close to zero or one. Strictly speaking, in computing Z we should use a continuity correction factor to transform the discrete binomial into a continuous normal distribution, that is

$$Z = \frac{\left(X \pm \frac{1}{2}\right) - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}} = \frac{\left(P \pm \frac{1}{2n}\right) - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$$

We should use a plus sign (+) when $X < n\,\pi_0$ or $P < \pi_0$ and a minus sign (−) when $X > n\,\pi_0$ or $P > \pi_0$.

The summary of the procedure of testing a hypothesis about the proportion of successes in a population, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing population proportion:
Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0 : \pi \geq \pi_0$ $H_1 : \pi < \pi_0$	$H_0 : \pi \leq \pi_0$ $H_1 : \pi > \pi_0$	$H_0 : \pi = \pi_0$ $H_1 : \pi \neq \pi_0$

Set level of significance: α

$$Z = \frac{P - \pi_0}{\sqrt{\pi_0 (1 - \pi_0)/n}}$$

$$P = \frac{X}{n}$$

Reject H_0 if $Z < z_\alpha$	Reject H_0 if $Z > z_{1-\alpha}$	Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

Calculate the observed value of test statistic

$$z = \frac{p - \pi_0}{\sqrt{\pi_0 (1 - \pi_0)/n}}$$

$$p = \frac{x}{n}$$

We reject H_0 if $z < z_\alpha$ otherwise we do not reject H_0	We reject H_0 if $z > z_{1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ otherwise we do not reject H_0

Example 13.15 *The reputations (and hence sales) of many businesses can be severely damaged by shipments of manufactured items that contain an unusually large percentage of defective items. A manufacturer of flashbulbs of cameras may want to be reasonable certain that less than 5% of its bulbs are defective. Suppose 300 bulbs are randomly selected from a very large shipment, each is tested, and 10 defective bulbs are found. Does this provide sufficient evidence for the manufacture to conclude that the fraction defective in the entire shipment is less than 0.05. Use* $\alpha = 0.01$.

Solution. We have $n = 300, \quad x = 10, \quad p = \frac{x}{n} = \frac{10}{300} = 0.033$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the proportion of defectives π is less than 0.05. Consequently, we will test the null hypothesis that $\pi \geq 0.05$ against the alternative hypothesis that $\pi < 0.05$.

The elements of the one-sided left tail test of hypothesis are:

Null hypothesis $H_0 : \pi \geq 0.05$

Alternative hypothesis $H_1 : \pi < 0.05$

Level of significance: $\alpha = 0.01$

Test statistic: $Z = \frac{P - \pi_0}{\sqrt{\pi_0 (1 - \pi_0)/n}}$ follows an approximate standard normal distribution under H_0

Critical value: $z_\alpha = z_{0.01} = -2.326$ { From Table 10 (*b*) }

Critical region: $Z < -2.326$

Decision rule: Reject H_0 if $Z < -2.326$, otherwise do not reject H_0.

Observed value: $z = \frac{p - \pi_0}{\sqrt{\pi_0 (1 - \pi_0)/n}} = \frac{0.033 - 0.05}{\sqrt{0.05 (1 - 0.05)/300}} = -1.351$

Conclusion: Since $z = -1.351 > -2.326$, we do not reject H_0. The manufacturer cannot conclude with 99% confidence that the shipment contains fewer than 5% defective bulbs.

Example 13.16 *It is known that approximately 10% smokers prefer cigarette brand A. After a promotional campaign in a given sales region, a sample of 200 cigarette smokers were interviewed to determine the effectiveness of the campaign. The results of the survey showed that 26 people expressed a preference of brand A. Do these data present a sufficient evidence to indicate an increase in the acceptance of brand A in the region. Use* $\alpha = 0.05$.

Solution. We have $n = 200, \quad x = 26, \quad p = \frac{x}{n} = \frac{26}{200} = 0.13$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the proportion of smokers π is greater than 0.10. Consequently, we will test the null hypothesis that $\pi \leq 0.10$ against the alternative hypothesis that $\pi > 0.10$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0 : \pi \leq 0.10$

Alternative hypothesis $H_1 : \pi > 0.10$

Level of significance: $\alpha = 0.05 \quad \Rightarrow \quad 1 - \alpha = 0.95$

Test statistic: $Z = \frac{P - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$ follows an approximate standard normal distribution under H_0

Critical value: $z_{1-\alpha} = z_{0.95} = 1.645$ { From Table 10 (*b*) }

Critical region: $Z > 1.645$

Decision rule: Reject H_0 if $Z > 1.645$, otherwise do not reject H_0.

Observed value: $z = \frac{p - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} = \frac{0.13 - 0.10}{\sqrt{0.10(1-0.10)/200}} = 1.414$

Conclusion: Since $z = 1.414 < 1.645$, so we do not reject H_0. We cannot conclude with 95% confidence that the promotional campaign is effective.

Example 13.17 *A supplier of components to a motor industry makes a particular product which sometimes fails immediately it is used. He controls his manufacturing process so that the proportion of faulty products is supposed to be only 4%. Out of 500 supplied in one batch 28 prove to be faulty. Has the process gone out of control to produce too many faulty products. Test at $\alpha = 0.05$ applying continuity correction.*

Solution. We have $n = 500$, $x = 28$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the proportion of faulty products π is greater than 0.04. Consequently, we will test the null hypothesis that $\pi \le 0.04$ against the alternative hypothesis that $\pi > 0.04$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \pi \le 0.04$

Alternative hypothesis $H_1: \pi > 0.04$

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $Z = \frac{(X \pm 0.5) - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}}$ follows an approximate standard normal distribution under H_0

Critical value: $z_{1-\alpha} = z_{0.95} = 1.645$ { From Table 10 (*b*) }

Critical region: $Z > 1.645$

Decision rule: Reject H_0 if $Z > 1.645$, otherwise do not reject H_0.

Observed value: $z = \frac{(x \pm 0.5) - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}} = \frac{(28 - 0.5) - 500(0.04)}{\sqrt{500(0.04)(1-0.04)}} = 1.71$

Conclusion: Since $z = 1.71 > 1.645$, we reject H_0 and conclude that the process is out of control.

Example 13.18 *The records of a certain hospital showed the birth of 723 males and 617 females in a certain week. Do these figures conform to the hypothesis that the sexes are born in equal proportions. Use $\alpha = 0.02$.*

Solution. We have $x = 723$, $n - x = 617$, $n = 1340$

$$p = \frac{x}{n} = \frac{723}{1340} = 0.54$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the proportion of male births π is not equal to 0.5. Consequently, we will test the null hypothesis that $\pi = 0.5$ against the alternative hypothesis that $\pi \ne 0.5$.

The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \pi = 0.5$

Alternative hypothesis $H_1: \pi \ne 0.5$

Level of significance: $\alpha = 0.02 \Rightarrow \alpha/2 = 0.01 \Rightarrow 1 - \alpha/2 = 0.99$

Test statistic: $Z = \frac{P - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$ follows an approximate standard normal distribution under H_0

Critical values: $z_{\alpha/2} = z_{0.01} = -2.326$,

$z_{1-\alpha/2} = z_{0.99} = 2.326$ { From Table 10 (*b*) }

Critical region: $Z < -2.326$ or $Z > 2.326$

Decision rule: Reject H_0 if $Z < -2.326$ or $Z > 2.326$, otherwise do not reject H_0

Observed value: $z = \frac{p - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} = \frac{0.54 - 0.5}{\sqrt{0.5(1-0.5)/1340}} = 2.928$

Conclusion: Since $z = 2.928 > 2.326$, we reject H_0 and conclude that that the proportion of male births is not equal to 0.5.

Exercise 13.3

1. (*a*) For each of the following sets of data, carry out a significance test for the hypotheses stated.

Sample	Number in sample	Number of successes	Hypotheses H_0	Hypotheses H_1	Level of significance
(*i*)	50	45	$\pi \le 0.8$	$\pi > 0.8$	5%
(*ii*)	60	42	$\pi = 0.55$	$\pi \ne 0.55$	2%
(*iii*)	120	21	$\pi = 1/4$	$\pi \ne 1/4$	5%
(*iv*)	300	213	$\pi = 0.65$	$\pi \ne 0.65$	1%
(*v*)	90	56	$\pi \ge 0.76$	$\pi < 0.76$	1%

[(*i*) Since $z = 1.768 > 1.645 = z_{0.95}$, we reject $H_0: \pi \le 0.8$ in favour of $H_1: \pi > 0.8$.

(*ii*) Since $z = 2.335 > 2.326 = z_{0.99}$, we reject $H_0: \pi = 0.55$ in favour of $H_1: \pi \ne 0.55$.

(iii) Since $z_{0.025} = -1.96 < z = 1.897 < 1.96 = z_{0.975}$, we do not reject $H_0: \pi = 1/4$ against $H_1: \pi \neq 1/4$.

(iv) Since $z_{0.005} = -2.576 < z = 2.179 < 2.576 = z_{0.995}$, we do not reject $H_0: \pi = 0.65$ against $H_1: \pi \neq 0.65$.

(v) Since $z = -3.060 < -2.326 = z_{0.01}$, we reject $H_0: \pi \geq 0.76$ in favour of $H_1: \pi < 0.76$.)

(b) A basket ball player has hit on 60% of his shots from the floor. If on the next 100 shots he makes 70 baskets, would you say that his shooting has improved. (Use a 0.05 level of significance).

(Since $z = 2.041 > 1.645 = z_{0.95}$, we reject $H_0: \pi \leq 0.60$ in favour of $H_1: \pi > 0.60$; Yes)

2. (a) In a poll of 10000 voters selected at a random from all the voters in a certain district, it is found that 5180 voters are in favour of a particular candidate. Test the null hypotheses that the proportion of all the voters in the district, who favour the candidate is equal to or less than 50% against the alternative that it is greater than 50%. Use a 0.05 level of significance.

(Since $z = 3.6 > 1.645 = z_{0.95}$, we reject $H_0: \pi \leq 0.5$ in favour of $H_1: \pi > 0.5$)

(b) A retailer places an order for 400 recapped automobile tires with a supplier who claims that no more than 5% of his output is ever returned unsatisfactory. In time, 31 of the 400 tires are unsatisfactory. Should the retailer continue to trust his supplier's word as to the rate of returns? Use 5% level of significance.

(Since $z = 2.524 > 1.645 = z_{0.95}$, we reject $H_0: \pi \leq 0.05$ in favour of $H_1: \pi > 0.05$; No)

(c) A commonly prescribed drug on a market for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension showed that 70 got relief. Is this sufficient evidence that the new drug is superior to the one commonly prescribed? Use 5% level of significance.

(Since $z = 2.041 > 1.645 = z_{0.95}$, we reject $H_0: \pi \leq 0.60$ in favour of $H_1: \pi > 0.60$; Yes)

3. (a) An electrical company claimed that at least 95% of the parts which they supplied on a government contract confirmed to specification. A sample of 400 parts was tested, and 355 meet specification. Can we accept the company's claim at a 0.05 level of significance?

(Since $z = -5.735 < -1.645 = z_{0.05}$, we reject $H_0: \pi \geq 0.95$ in favour of $H_1: \pi < 0.95$; No)

(b) An electric company claimed that at least 85% of the parts which they supplied conformed to specifications. A sample of 400 parts was tested and 75 did not meet specifications. Can we accept the company's claim at 0.05 level of significance?

(Since $z = -2.100 < -1.645 = z_{0.05}$, we reject $H_0: \pi \geq 0.85$ in favour of $H_1: \pi < 0.85$; No)

(c) The manufacturer of a patent medicine claimed that it was at least 90% effective in relieving an allergy for a period of 8 hours. In a sample of 200 people, who had the allergy, the medicine provided relief for 160 people. Determine whether the manufacturer's claim is legitimate.

(Since $z = -4.714 < -1.645 = z_{0.05}$, we reject $H_0: \pi \geq 0.90$ in favour of $H_1: \pi < 0.90$; No)

4. (a) A coin is tossed 100 times and 38 heads are obtained. Is there evidence, at the 2% level that the coin is biased in favour of tails?

(Since $z = -2.4 < -2.05 = z_{0.02}$, we reject $H_0: \pi \geq 0.5$ in favour of $H_1: \pi < 0.5$; Yes)

(b) A coin is tossed 400 times and it turns up heads 216 times. Discuss whether the coin may be an unbiased one.

(Since $z_{0.25} = -1.960 < z = 1.6 < 1.960 = z_{0.975}$, we do not reject $H_0: \pi = 0.5$ against $H_1: \pi \neq 0.5$)

5. (a) In a random sample of 1000 houses in a certain city, 618 own colour TV sets. Is this sufficient evidence to conclude that 2/3 of the houses in this city have colour TV sets? Use $\alpha = 0.02$.

(Since $z = -3.288 < -2.326 = z_{0.01}$, we reject $H_0: \pi = 2/3$ in favour of $H_1: \pi \neq 2/3$; No)

(b) The sex distribution of 98 births reported in a newspaper was 52 boys and 46 girls. Is this consistent with an equal sex division in the population? Use 5% level of significance.

(Since $z_{0.25} = -1.960 < z = 0.594 < 1.960 = z_{0.975}$, we do not reject $H_0: \pi = 0.5$ against $H_1: \pi \neq 0.5$.)

6. (a) The reputations (and hence sales) of many businesses can be severely damaged by shipments of manufactured items that contain an unusually large percentage of defective items. A manufacturer of flashbulbs of cameras may want to be reasonable certain that less than 5% of its bulbs are defective. Suppose 300 bulbs are randomly selected from a very large shipment, each is tested, and 90 good bulbs are found. Does this provide sufficient evidence for the manufacture to conclude that the fraction defective in the entire shipment is less than 0.05. Use $\alpha = 0.01$.

(Since $z = -1.351 > -2.326 = z_{0.01}$, we do not reject H_0. The manufacturer cannot conclude with 99% confidence that the shipment contains fewer than 5% defective bulbs)

(b) A supplier of components to a motor industry makes a particular product which sometimes fails immediately it is used. He controls his manufacturing process so that the proportion of faulty products is supposed to be only 4%. Out of 500 supplied in one batch 28 prove to be faulty. Has the process gone out of control to produce too many faulty products. Test at $\alpha = 0.05$ applying continuity correction.

(Since $z = 1.71 > 1.645 = z_{0.95}$, we reject $H_0: \pi \leq 0.04$ in favour of $H_1: \pi > 0.04$ and conclude that the process is out of control)

13.4 TEST OF HYPOTHESES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION MEANS — *INDEPENDENT SAMPLES*

There are many problems where we are interested in hypotheses concerning about the differences between the means of two populations. For instance, we may wish to decide upon the basis of suitable samples whether a new fertilizer is more effective than an existing fertilizer, or whether a newly introduced product is more reliable than an existing product. Specifically, within the frame work of statistical language, we are interested in making inferences about the parameter $\mu_1 - \mu_2$. A test of hypothesis must be based on assumptions regarding the structure of the underlying distributions. Two independent random samples must be taken — one from each of the two populations of interest.

13.4.1 Forms of Hypothesis. We are interested in tests about the parameter $\mu_1 - \mu_2$. Let δ_0 be a hypothesized value of the difference between two population means the three possible null hypotheses about the difference between two population means, and their corresponding alternative hypotheses, are:

1. $H_0: \mu_1 - \mu_2 \geq \delta_0$ against $H_1: \mu_1 - \mu_2 < \delta_0$
2. $H_0: \mu_1 - \mu_2 \leq \delta_0$ against $H_1: \mu_1 - \mu_2 > \delta_0$
3. $H_0: \mu_1 - \mu_2 = \delta_0$ against $H_1: \mu_1 - \mu_2 \neq \delta_0$

13.4.2 Independent Samples: ***Normal populations, known variances, any sample sizes.*** Suppose that we have two independent random samples of sizes n_1 and n_2 from two normal populations having, respectively, means μ_1 and μ_2 and known variances σ_1^2 and σ_2^2. We known that $\overline{X}_1$ is distributed as $N(\mu_1, \sigma_1^2/n_1)$ and that $\overline{X}_2$ is distributed as $N(\mu_2, \sigma_2^2/n_2)$, and that $\overline{X}_1$ is independent of $\overline{X}_2$. The statistic $\overline{X}_1 - \overline{X}_2$, the difference between two independent normal variables, is also normally distributed with mean $\mu_1 - \mu_2$ and variance $\sigma_1^2/n_1 + \sigma_2^2/n_2$. Consequently the random variable

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

has a standard normal distribution. Thus the appropriate test statistic is

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - \delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

13.4.3 Independent Samples: ***Normal populations, same unknown variance, small samples.*** Suppose that we have two independent random samples of sizes n_1 and n_2 ($n_1 < 30$ and $n_2 < 30$) from two normal populations having, respectively, means μ_1 and μ_2 and unknown

common variance σ^2 (*i. e.,* $\sigma_1^2 = \sigma_2^2 = \sigma^2$). Then $\overline{X}_1$ is normally distributed with mean μ_1 and variance σ^2/n_1 and $\overline{X}_2$ is normally distributed with mean μ_2 and variance σ^2/n_2 and that $\overline{X}_1$ is independent of $\overline{X}_2$.

Therefore, $\overline{X}_1 - \overline{X}_2$ is normally distributed with mean

$$\mu_{\overline{X}_1 - \overline{X}_2} = \mu_1 - \mu_2$$

and variance

$$\sigma^2_{\overline{X}_1 - \overline{X}_2} = \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2} = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Thus the random variable

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

has a standard normal distribution. Since we assume that the two populations have equal unknown variances ($\sigma_1^2 = \sigma_2^2 = \sigma^2$ unknown), we will replace the population variance by the sample variance.

The difficulty in making this replacement lies in the fact that we have two estimates of σ^2, $\hat{S}_1^2$ and $\hat{S}_2^2$, since two different samples were collected. Even if $\sigma_1^2 = \sigma_2^2 = \sigma^2$, it is unlikely that the two samples collected will have exactly the same value because of sampling error. But if $\hat{S}_1^2$ and $\hat{S}_2^2$ differ which of these two estimates should be used to estimate the unknown population variance σ^2.

Since we wish to obtain the best estimate available, it would seem reasonable to use an estimator that would pool the information from both samples. Thus, if $\hat{S}_1^2$ and $\hat{S}_2^2$ are the two sample variances (both estimating the variance σ^2 common to both populations), the pooled (weighted arithmetic mean) estimator of σ^2, denoted by S_p^2, is

$$S_p^2 = \frac{(n_1 - 1)\hat{S}_1^2 + (n_2 - 1)\hat{S}_2^2}{n_1 + n_2 - 2}$$

$$= \frac{\Sigma(X_{i1} - \overline{X}_1)^2 + \Sigma(X_{i2} - \overline{X}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{(\Sigma X_{i1}^2 - n_1\overline{X}_1^2) + (\Sigma X_{i2}^2 - n_2\overline{X}_2^2)}{n_1 + n_2 - 2}$$

To obtain the small-samples test statistic for testing $H_0: \mu_1 - \mu_2 = \delta_0$, substitute the pooled estimator of σ^2 into the above formula to obtain

$$T = \frac{(\overline{X}_1 - \overline{X}_2) - \delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

which has a t-distribution with $v = n_1 + n_2 - 2$ degrees of freedom.

13.4.4 Independent Samples: ***Any populations, large samples.*** When both sample sizes are large (say greater than 30) the assumptions regarding small samples can be greatly relaxed. It is no longer necessary to assume that the parent distributions are normal, because the Central Limit Theorem assures that $\overline{X}_1$ is approximately normally distributed with mean μ_1 and variance σ_1^2/n_1, and that $\overline{X}_2$ is also approximately normally distributed with mean μ_2 and variance σ_2^2/n_2, and that $\overline{X}_1$ is independent of $\overline{X}_2$, then $\overline{X}_1 - \overline{X}_2$ is approximately normally distributed with mean $\mu_1 - \mu_2$ and variance $\sigma_1^2/n_1 + \sigma_2^2/n_2$.

Thus the random variable

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}}}$$

has an approximate standard normal distribution.

Because n_1 and n_2 are both large, the approximation remains valid if σ_1^2 and σ_2^2 are replaced by their sample variances $\hat{S}_1^2$ and $\hat{S}_2^2$. The assumption of equal variance is not required in inferences derived from large samples. We can still use Z test with $\hat{S}_1^2$ substituted for σ_1^2 and $\hat{S}_2^2$ substituted for σ_2^2 so long as both samples are large enough for the Central Limit Theorem to be applied.

That is, we use

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - \delta_0}{\sqrt{\frac{\hat{S}_1^2}{n_1} + \frac{\hat{S}_2^2}{n_2}}}$$

as the test statistic.

The summary of the procedure of testing a hypothesis about the difference between means of two populations whose variances σ_1^2, σ_2^2 are known/unknown, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing difference between means of two populations, σ_1^2, σ_2^2 known/unknown:

Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \mu_1 - \mu_2 \geq \delta_0$ $H_1: \mu_1 - \mu_2 < \delta_0$	$H_0: \mu_1 - \mu_2 \leq \delta_0$ $H_1: \mu_1 - \mu_2 > \delta_0$	$H_0: \mu_1 - \mu_2 = \delta_0$ $H_1: \mu_1 - \mu_2 \neq \delta_0$

Set level of significance: α

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - \delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \equiv \frac{(\overline{X}_1 - \overline{X}_2) - \delta_0}{\sqrt{\frac{\hat{S}_1^2}{n_1} + \frac{\hat{S}_2^2}{n_2}}}$$

Reject H_0 if $Z < z_\alpha$	Reject H_0 if $Z > z_{1-\alpha}$	Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

Calculate the observed value of test statistic

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \equiv \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{\frac{\hat{s}_1^2}{n_1} + \frac{\hat{s}_2^2}{n_2}}}$$

We reject H_0 if $z < z_\alpha$ otherwise we do not reject H_0	We reject H_0 if $z > z_{1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ otherwise we do not reject H_0

The summary of the procedure of testing a hypothesis about the difference between means of two normal populations with same unknown variance $\sigma_1^2 = \sigma_2^2 = \sigma^2$, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing difference between means of two normal populations, same unknown variance: Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \mu_1 - \mu_2 \geq \delta_0$ $H_1: \mu_1 - \mu_2 < \delta_0$	$H_0: \mu_1 - \mu_2 \leq \delta_0$ $H_1: \mu_1 - \mu_2 > \delta_0$	$H_0: \mu_1 - \mu_2 = \delta_0$ $H_1: \mu_1 - \mu_2 \neq \delta_0$

Set level of significance: α

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - \delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S_p = \sqrt{\frac{\Sigma(X_{i1} - \bar{X}_1)^2 + \Sigma(X_{i2} - \bar{X}_2)^2}{n_1 + n_2 - 2}}$$

Reject H_0 if $T < t_{v;\alpha}$ $v = n_1 + n_2 - 2$	Reject H_0 if $T > t_{v;1-\alpha}$	Reject H_0 if $T < t_{v;\alpha/2}$ or $T > t_{v;1-\alpha/2}$

Calculate the observed value of test statistic

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$s_p = \sqrt{\frac{\Sigma(x_{i1} - \bar{x}_1)^2 + \Sigma(x_{i2} - \bar{x}_2)^2}{n_1 + n_2 - 2}}$$

We reject H_0 if $t < t_{v:\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t > t_{v:1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t < t_{v:\alpha/2}$ or $t > t_{v:\alpha/2}$ otherwise we do not reject H_0

***Example* 13.19** *Apex's current packaging machinery for coffee is known to ground coffee into "1-pound cans" with a standard deviation of 0.6 ounce. Apex is considering using a new packaging machine which is expected to pour coffee into "1-pound cans" more accurately, with a standard deviation of 0.3 ounce. Before deciding to invest in the new machine, Apex wished to test its performance against the old machine. A sample was taken on each machine to measure the mean weight of contents of the "1-pound can", with the following results.*

Sample	Size	Mean
Using Old Machine	$n_1 = 25$	$\bar{x}_1 = 16.7$ ounces
Using New Machine	$n_2 = 36$	$\bar{x}_2 = 15.8$ ounces

Test, at the 5% level of significance, the hypothesis that there is no difference in the average weight of the contents poured by the old machine versus the new machine.

Solution. We have $n_1 = 25$, $\bar{x}_1 = 16.7$, $\sigma_1 = 0.6$

$n_2 = 36$, $\bar{x}_2 = 15.8$, $\sigma_2 = 0.3$

The objective of the **sampling** is to attempt to support the research (alternative) hypothesis that the average weight of the contents poured by the old machine μ_1 differs from that of the new machine μ_2. Consequently, we will test the null hypothesis that $\mu_1 = \mu_2$ against the alternative hypothesis that $\mu_1 \neq \mu_2$.

The elements of the one-sided right tail test of hypothesis are:

The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \mu_1 = \mu_2 \Rightarrow \mu_1 - \mu_2 = 0$ (*i. e.* $\delta_0 = 0$)

Alternative hypothesis $H_1: \mu_1 \neq \mu_2 \Rightarrow \mu_1 - \mu_2 \neq 0$ (*i. e.* $\delta_0 \neq 0$)

Level of significance: $\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow 1 - \alpha/2 = 0.975$

Test statistic: $Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - \delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$ follows a standard normal distribution under H_0

Critical values: $z_{\alpha/2} = z_{0.025} = -1.960$,

$z_{1-\alpha/2} = z_{0.975} = 1.960$ { From Table 10 (*b*) }

Critical region: $Z < -1.960$ or $Z > 1.960$

Decision rule: Reject H_0 if $Z < -1.960$ or $Z > 1.960$, otherwise do not reject H_0.

Observed value: $z = \dfrac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} = \dfrac{(16.7 - 15.8) - 0}{\sqrt{\dfrac{(0.6)^2}{25} + \dfrac{(0.3)^2}{36}}} = 6.92$

Conclusion: **Since $z = 6.92 > 1.960$, we reject H_0 and conclude that the mean weight of a "1-pound can" filled by the new machine is not the same as that of filled by the old machine.**

***Example* 13.20** *The test was given to a group of 100 scouts and to a group of 144 guides. The mean score for the scouts was 27.53 and the mean score for the guides was 26.81.*

Assuming a common population standard deviation of 3.48, test, using a 5% level of significance, whether the scouts performance in the test was better than that of the guides.

Solution. We have $n_1 = 100, \quad \bar{x}_1 = 27.53$

$n_2 = 144, \quad \bar{x}_2 = 26.81$

Common population standard deviation: $\sigma = 3.48$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean score for the scouts μ_1 is greater than that of the guides μ_2. Consequently, we will test the null hypothesis that $\mu_1 \le \mu_2$ against the alternative hypothesis that $\mu_1 > \mu_2$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu_1 \le \mu_2 \Rightarrow \mu_1 - \mu_2 \le 0$ (*i. e.* $\delta_0 \le 0$)

Alternative hypothesis $H_1: \mu_1 > \mu_2 \Rightarrow \mu_1 - \mu_2 > 0$ (*i. e.* $\delta_0 > 0$)

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - \delta_0}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$ follows a standard normal distribution under H_0

Critical value: $z_{1-\alpha} = z_{0.95} = 1.645$ { From Table 10 (*b*) }

Critical region: $Z > 1.645$

Decision rule: Reject H_0 if $Z > 1.645$, otherwise do not reject H_0.

Observed value: $$z = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(27.53 - 26.81) - 0}{3.48\sqrt{\frac{1}{100} + \frac{1}{144}}} = 1.589$$

Conclusion: Since $z = 1.589 < 1.645$, we do not reject H_0 and conclude that there is not sufficient evidence, at 5% level of significance, to show that the performance of the scouts in the test was better than that of the guides.

Example 13.21 *The management of a restaurant wants to determine whether a new advertising campaign has increased its mean daily income (gross). The income for 50 business days prior to the campaign's beginning were recorded. After conducting the advertising campaign and allowing a 20 days period for the advertising to take effect, the restaurant management recorded the income for 30 business days. These two samples will allow the management to make an inference about the effect of the advertising campaign on the restaurant's daily income. A summary of the results of the two samples are shown below:*

Sample	*Size*	*Mean*	*Standard deviation*
Before campaign	$n_1 = 50$	$\bar{x}_1 = 1255$	$\hat{s}_1 = 215$
After campaign	$n_2 = 30$	$\bar{x}_2 = 1330$	$\hat{s}_2 = 238$

Do these samples provide sufficient evidence for the management to conclude that the mean income has been increased by the advertising campaign test using $\alpha = 0.05$.

Solution. The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean daily income after advertising campaign μ_2 is greater than that of before the campaign μ_1. Consequently, we will test the null hypothesis that $\mu_2 \le \mu_1$ against the alternative hypothesis that $\mu_2 > \mu_1$.

The elements of the one-sided right tail test of hypothesis are

Null hypothesis $H_0: \mu_2 \le \mu_1 \Rightarrow \mu_2 - \mu_1 \le 0$ (*i. e.* $\delta_0 \le 0$)

Alternative hypothesis $H_1: \mu_2 > \mu_1 \Rightarrow \mu_2 - \mu_1 > 0$ (*i. e.* $\delta_0 > 0$)

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $$Z = \frac{(\bar{X}_2 - \bar{X}_1) - \delta_0}{\sqrt{\frac{\hat{S}_1^2}{n_1} + \frac{\hat{S}_2^2}{n_2}}}$$ follows a standard normal distribution under H_0

Critical value: $z_{1-\alpha} = z_{0.95} = 1.645$ { From Table 10 (*b*) }

Critical region: $Z > 1.645$

Decision rule: Reject H_0 if $Z > 1.645$, otherwise do not reject H_0

Observed value: $$z = \frac{(\bar{x}_2 - \bar{x}_1) - \delta_0}{\sqrt{\frac{\hat{s}_1^2}{n_1} + \frac{\hat{s}_2^2}{n_2}}} = \frac{(1330 - 1255) - 0}{\sqrt{\frac{(215)^2}{50} + \frac{(238)^2}{30}}} = 1.414$$

Conclusion: Since $z = 1.414 < 1.645$, we do not reject H_0. That is, the samples do not provide sufficient evidence, at the $\alpha = 0.05$ significance level, for the restaurant management to conclude that the advertising campaign has increased the mean daily income.

Example 13.22 *A feeding test is conducted on a herd of 25 milking cows to compare two diets, one of dewatered alfalfa and the other of field-wilted alfalfa. Dewatered alfalfa has an economic advantage in that its mechanical processing produces a liquid protein-rich by product that can be used to supplement the feed of other animals. A sample of 12 cows randomly selected from the herd are fed dewatered alfalfa; the remaining 13 cows are fed field-wilted alfalfa. From observations made over a three-week period, the average daily milk production in pounds recorded for each cow is:*

Field-wilted alfalfa	44	44	56	46	47	38	58	53	49	35	46	30	41
Dewatered alfalfa	35	47	55	29	40	39	32	41	42	57	51	39	

Do the data strongly indicate that the milk yield is less with dewatered alfalfa than with field-wilted alfalfa? Test at $\alpha = 0.05$.

Solution. The means of two samples and estimate of common variance are

x_1	44	44	56	46	47	38	58	53	49	35	46	30	41	$\sum x_1 = 587$
x_2	35	47	55	29	40	39	32	41	42	57	51	39		$\sum x_2 = 507$
x_1^2	1936	1936	3136	2116	2209	1444	3364	2809	2401	1225	2116	900	1681	$\sum x_1^2 = 27273$
x_2^2	1225	2209	3025	841	1600	1521	1024	1681	1764	3249	2601	1521		$\sum x_2^2 = 22261$

$$\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{587}{13} = 45.15$$

$$\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{507}{12} = 42.25$$

$$\sum(x_1 - \bar{x}_1)^2 = \sum x_1^2 - \frac{(\sum x_1)^2}{n_1} = 27273 - \frac{(587)^2}{13} = 767.69$$

$$\sum(x_2 - \bar{x}_2)^2 = \sum x_2^2 - \frac{(\sum x_2)^2}{n_2} = 22261 - \frac{(507)^2}{12} = 840.25$$

$$s_p = \sqrt{\frac{\sum(x_1-\bar{x}_1)^2 + \sum(x_2-\bar{x}_2)^2}{n_1+n_2-2}} = \sqrt{\frac{767.69 + 840.25}{13 + 12 - 2}} = 8.36$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean milk yield with dewatered alfalfa μ_2 is less than with field-wilted alfalfa μ_1. Consequently, we will test the null hypothesis that $\mu_2 \geq \mu_1$ against the alternative hypothesis that $\mu_2 < \mu_1$.

The elements of the one-sided left tail test of hypothesis are:

Null hypothesis $H_0: \mu_2 \geq \mu_1 \Rightarrow \mu_2 - \mu_1 \geq 0$ (i. e. $\delta_0 \geq 0$)

Alternative hypothesis $H_1: \mu_2 < \mu_1 \Rightarrow \mu_2 - \mu_1 < 0$ (i. e. $\delta_0 < 0$)

Level of significance: $\alpha = 0.05$

Test statistic: $T = \dfrac{(\bar{X}_2 - \bar{X}_1) - \delta_0}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n_1 + n_2 - 2 = 13 + 12 - 2 = 23$

Critical value: $t_{\nu;\alpha} = t_{23;0.05} = -1.714$ (From Table 12)

Critical region: $T < -1.714$

Decision rule: Reject H_0 if $T < -1.714$, otherwise do not reject H_0

Observed value: $t = \dfrac{(\bar{x}_2 - \bar{x}_1) - \delta_0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{(42.25 - 45.15) - 0}{8.36\sqrt{\dfrac{1}{13} + \dfrac{1}{12}}} = -0.87$

Conclusion: Since $t = -0.87 > -1.714$, we do not reject $H_0: \mu_2 \geq \mu_1$ against $H_1: \mu_2 < \mu_1$.

Example **13.23** *Two random samples taken independently from normal populations with an identical variance yield the following results.*

Sample	Size	Mean	Variance
I	$n_1 = 12$	$\bar{x}_1 = 10$	$\hat{s}_1^2 = 1200$
II	$n_2 = 18$	$\bar{x}_2 = 25$	$\hat{s}_2^2 = 900$

Test the hypotheses that the true difference between the population means is at most 10, that is $\mu_2 - \mu_1 \leq 10$, against the alternative that $\mu_2 - \mu_1 > 10$ at 5% level of significance.

Solution. The estimate of common variance is

$$s_p = \sqrt{\frac{(n_1-1)\hat{s}_1^2 + (n_2-1)\hat{s}_2^2}{n_1+n_2-2}}$$

$$= \sqrt{\frac{(12-1)1200 + (18-1)900}{12 + 18 - 2}} = 31.9$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the mean of second population μ_2 is greater than that of first population μ_1 by more than 10 points. Consequently, we will test the null hypothesis that $\mu_2 - \mu_1 \leq 10$ against the alternative hypothesis that $\mu_2 - \mu_1 > 10$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu_2 - \mu_1 \leq 10$ (i. e. $\delta_0 \leq 10$)

Alternative hypothesis $H_1: \mu_2 - \mu_1 > 10$ (i. e. $\delta_0 > 10$)

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $T = \dfrac{(\bar{X}_2 - \bar{X}_1) - \delta_0}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n_1 + n_2 - 2 = 12 + 18 - 2 = 28$

Critical values: $t_{\nu;1-\alpha} = t_{28;0.95} = 1.701$ (From Table 12)

Critical region: $T > 1.701$

Decision rule: Reject H_0 if $T > 1.701$, otherwise do not reject H_0.

Observed value: $t = \dfrac{(\bar{x}_2 - \bar{x}_1) - \delta_0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{(25 - 10) - 10}{31.9\sqrt{\dfrac{1}{12} + \dfrac{1}{18}}} = 0.421$

Conclusion: Since $t = 0.421 < 1.701 = t_{28;0.95}$, we do not reject H_0.

Exercise 13.4

1. (a) For each of the following sets of data, perform a test to decide whether there is a significant difference between the means, μ_1 and μ_2, of the normal populations from which the samples are drawn.

	n_1	$\sum x_1$	σ_1^2	n_2	$\sum x_2$	σ_2^2	Hypotheses	α
(i)	100	4250	30	80	3544	35	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	5%
(ii)	20	95	2.3	25	135	2.5	$H_0: \mu_1 \geq \mu_2$ $H_1: \mu_1 < \mu_2$	2%
(iii)	50	1545	6.5	50	1480	7.1	$H_0: \mu_1 \leq \mu_2$ $H_1: \mu_1 > \mu_2$	1%
	n_1	$\sum x_1$	n_2	$\sum x_2$	Common Population standard deviation σ		Hypotheses	α
(iv)	100	12730	100	12410	10.9		$H_0: \mu_1 \leq \mu_2$ $H_1: \mu_1 > \mu_2$	5%
(v)	30	192	45	315	1.25		$H_0: \mu_1 \geq \mu_2$ $H_1: \mu_1 < \mu_2$	1%
(vi)	200	18470	300	27663	0.86		$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	10%
	n_1	$\sum x_1$	$\sum (x_1 - \bar{x}_1)^2$	n_2	$\sum x_2$	$\sum (x_2 - \bar{x}_2)^2$	Hypotheses	α
(vii)	40	2128	810	50	2580	772	$H_0: \mu_1 \leq \mu_2$ $H_1: \mu_1 > \mu_2$	5%
(viii)	80	6824	2508	100	8740	3969	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	2%
(ix)	65	5369	8886	80	4672	5026	$H_0: \mu_1 - \mu_2 \leq 20$ $H_1: \mu_1 - \mu_2 > 20$	1%

{ (i) Since $z = -2.096 < -1.96 = z_{0.025}$, we reject $H_0: \mu_1 = \mu_2$ in favour of $H_1: \mu_1 \neq \mu_2$.

(ii) Since $z = -1.402 > -2.054 = z_{0.02}$, we do not reject $H_0: \mu_1 \geq \mu_2$ against $H_1: \mu_1 < \mu_2$.

(iii) Since $z = 2.493 > 2.326 = z_{0.99}$, we reject $H_0: \mu_1 \leq \mu_2$ in favour of $H_1: \mu_1 > \mu_2$.

(iv) Since $z = 2.076 > 1.645 = z_{0.95}$, we reject $H_0: \mu_1 = \mu_2$ in favour of $H_1: \mu_1 > \mu_2$.

(v) Since $z = -2.036 > -2.326 = z_{0.01}$, we do not reject $H_0: \mu_1 \geq \mu_2$ against $H_1: \mu_1 < \mu_2$.

(vi) Since $z = 1.783 > 1.645 = z_{0.95}$, we reject $H_0: \mu_1 = \mu_2$ in favour of $H_1: \mu_1 \neq \mu_2$.

(vii) Since $z = 1.752 > 1.645 = z_{0.95}$, we reject $H_0: \mu_1 \leq \mu_2$ in favour of $H_1: \mu_1 > \mu_2$.

(viii) Since $z_{0.01} = -2.326 < z = -2.351 < 2.326 = z_{0.99}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$.

(ix) Since $z = 2.453 > 2.326 = z_{0.99}$, we reject $H_0: \mu_1 - \mu_2 \leq 20$ in favour of $H_1: \mu_1 - \mu_2 > 20$.}

(b) A simple sample of heights of 6400 Englishmen has a mean of 67.85 inches and a standard deviation of 2.56 inches, while a simple sample of heights of 6400 Australian has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that Australians are on the average taller than the Englishmen.
{ Since $z = 15.589 > 1.645 = z_{0.95}$, we reject $H_0: \mu_2 \leq \mu_1$ against $H_1: \mu_2 > \mu_1$ at $\alpha = 0.05$. }

2. (a) A random sample of 100 professors in private colleges showed an average monthly salary of Rs. 5000 with a standard deviation of Rs. 200. Another random sample of 150 professors in Govt. Colleges showed an average monthly salary of Rs. 5600 with a standard deviation of Rs. 250. Test the hypotheses that the average salary for the professors teaching in Govt. Colleges does not exceed the average salary for professors teaching in private colleges by more than Rs. 500. Use $\alpha = 0.01$.
{ Since $z = 3.499 > 2.326 = z_{0.99}$, we reject $H_0: \mu_2 - \mu_1 \leq 500$ in favour of $H_1: \mu_2 - \mu_1 > 500$. }

(b) A random sample of 80 light bulbs manufactured by company A had an average lifetime of 1258 hours with a standard deviation of 94 hours, while a random sample of 60 light bulbs manufactured by company B had an average lifetime of 1029 hours with a standard deviation of 68 hours. Because of the high cost of bulbs from company A, we are inclined to buy from company B unless the bulbs from company A will last over 200 hours longer on the average than those from company B. Run a test using $\alpha = 0.01$ to determine from whom we should buy our bulbs.
{ Since $z = 2.118 < 2.326 = z_{0.99}$, we do not reject $H_0: \mu_1 - \mu_2 \leq 200$ against $H_1: \mu_1 - \mu_2 > 200$. }

3. (a) A farmer claims the average yield of corn of variety A exceeds the average yield of variety B by at least 12 bushels per acre. To test this claim, 50 acres of each variety are planted and grown under similar conditions. Variety A yields on the average 86.7 bushels per acre with a standard deviation of 6.28 bushels per acre, while Variety B yields on the average 77.8 bushels per acre with a standard deviation of 5.61 bushels per acre. Test the farmer's claim using a 0.05 level of significance.
{ Since $z = -2.603 < -1.645 = z_{0.05}$, we reject $H_0: \mu_1 - \mu_2 \geq 12$ in favour of $H_1: \mu_1 - \mu_2 < 12$. }

(b) An examination was given to two classes of 40 and 50 students respectively. In the first class, mean grade was 74 with a standard deviation of 8, while in the second class the mean grade was 78 with a standard deviation of 7. Is there a significance difference between the mean grades at 1% level.

{ Since $z_{0.005} = -2.576 < z = 2.490 < 2.576 = z_{0.995}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. }

4. (a) A random sample of 100 light bulbs manufactured by company A had a mean lifetime of 1230 hours with a standard deviation of 80 hours, while a random sample of 121 light bulbs manufactured by company B had a mean lifetime of 1200 hours with a standard deviation of 66 hours. Are the mean lifetimes of bulbs manufactured by two companies are significantly different.

{ Since $z = 3 > 1.960 = z_{0.975}$, we reject $H_0: \mu_1 = \mu_2$ in favour of $H_1: \mu_1 \neq \mu_2$. }

(b) A random sample of 200 villages was taken from Faisalabad District and average population per village was found to be 498 with a standard deviation of 50. Another sample of 200 villages from the same district gave an average population 510 per village with a standard deviation of 40. Is the difference between the average of the two samples statistically significant?

{ Since $z = -2.650 < -1.960 = z_{0.025}$, we reject $H_0: \mu_1 = \mu_2$ in favour of $H_1: \mu_1 \neq \mu_2$. }

(c) The means of simple samples of 500 and 400 are 11.5 and 10.9 respectively. Can the samples be regarded as drawn from a population of standard deviation 5?

{ Since $z_{0.025} = -1.960 < z = 1.789 < 1.960 = z_{0.975}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. }

5. (a) Describe the procedure for testing the equality of means of two normal populations for:

(i) Large samples,

(ii) Small samples

(b) For each of the following sets of data, perform a test to decide whether there is a significant difference between the means, μ_1 and μ_2, of the normal populations from which the samples are drawn.

	n_1	$\sum x_1$	$\sum(x_1-\bar{x}_1)^2$	n_2	$\sum x_2$	$\sum(x_2-\bar{x}_2)^2$	Hypotheses	α
(i)	6	171	83	7	164.5	112	$H_0: \mu_1 \le \mu_2$ $H_1: \mu_1 > \mu_2$	5%
(ii)	5	678.5	562.3	7	971.6	308.6	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	5%
(iii)	8	238.4	296	10	206	145	$H_0: \mu_1 - \mu_2 \le 4$ $H_1: \mu_1 - \mu_2 > 4$	1%
(iv)	12	116.16	45.1	18	156.96	72	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	10%

{ (i) Since $t = 2.135 > 1.796 = t_{11;\,0.95}$, we reject $H_0: \mu_1 \le \mu_2$ in favour of $H_1: \mu_1 > \mu_2$

(ii) Since $t_{10;\,0.025} = -2.228 < t = -0.567 < 2.228 = t_{10;\,0.975}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$.

(iii) Since $t = 2.088 < 2.583 = t_{16;\,0.99}$, we do not reject $H_0: \mu_1 - \mu_2 \le 4$ against $H_1: \mu_1 - \mu_2 > 4$.

(iv) Since $t_{28;\,0.05} = -1.701 < t = 1.260 < 1.701 = t_{28;\,0.95}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. }

(c) Eight pots, growing three barley plants each, were exposed to a high tension discharge while nine similar pots were enclosed in an earthen wire cage, the number of tillers (shoots) in each pot were as follows:

Caged	17	18	25	27	29	27	23	17	28
Electrified	16	16	20	16	21	17	15	20	

Discuss whether the electrification exercises any real effect on tillering.

{ Since $t = 3.058 > 1.753 = t_{15;\,0.95}$, we reject $H_0: \mu_1 \le \mu_2$ in favour of $H_1: \mu_1 > \mu_2$ at $\alpha = 0.05$. }

6. (a) A random sample of 6 cows of breed A had daily milk yields in lb., as 16, 15, 18, 17, 19 and 17 and another random sample of 8 cows of breed B had daily milk yields in lb., as 18, 22, 21, 23, 19, 20, 24 and 21. Test if breed B is better than breed A at $\alpha = 0.05$.

{ Since $t = 4.162 > 1.782 = t_{12;\,0.95}$, we reject $H_0: \mu_2 \le \mu_1$ in favour of $H_1: \mu_2 > \mu_1$. }

(b) The heights of six randomly selected sailors are in inches: 62, 64, 67, 68, 70 and 71. Those of ten randomly selected soldiers are 62, 63, 65, 66, 69, 69, 70, 71, 72 and 73. Discuss in the light of these data that soldiers are on the average taller than sailors. Assume that the heights are normally distributed.

{ Since $t = 0.526 < 1.761 = t_{14;\,0.95}$, we do not reject $H_0: \mu_2 \le \mu_1$ against $H_1: \mu_2 > \mu_1$ at $\alpha = 0.05$. }

7. (a) The weights in grams of 10 male and 10 female juvenile ring-necked pheasants are:

Males	1293	1380	1614	1497	1340	1643	1466	1627	1383	1711
Females	1061	1065	1092	1017	1021	1138	1143	1094	1270	1028

Test the hypothesis of a difference of 350 grams between population means in favour of males against the alternative of a greater difference.

{ Since $t = 1.007 < 1.734 = t_{18;\,0.95}$, we do not reject $H_0: \mu_1 - \mu_2 \le 350$ against $H_1: \mu_1 - \mu_2 > 350$ at $\alpha = 0.05$. }

(b) The following data are the gains in weight, measured in pounds, of babies from birth to age six months. All babies in both groups weighed approximately the same at birth. The babies in sample I were fed formula A, and babies in sample II were fed formula B.

(Assume that the experimenter has no preconceived notions about which formula might be better).

Sample I	5	7	8	9	6	7	10	8	6
Sample II	9	10	8	6	8	7	9		

Test at the 5% level of significance that the mean of population I equals mean of population II.

{ Since $t_{14;\,0.025} = -2.145 < t = -1.083 < 2.145 = t_{14;\,0.975}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. }

8. (*a*) From the area planted in one variety of value, 54 plants were selected at random. Of these plants, 15 were "Off-types" and 12 were "Abberrant". The rubber percentages for these plants were:

Off-types	6.21,	5.70,	6.04,	4.47,	5.22,	4.45,	4.84,	5.88
	5.82,	6.09,	5.59,	6.06,	5.59,	6.74,	5.55.	
Aberrants	4.28,	7.71,	6.48,	7.71,	7.37,	7.20,	7.06,	6.40,
	8.93,	5.91,	5.51,	6.36.				

Test the hypothesis of no difference between the two means. Also compute a 95% confidence interval for the difference of two population means.

{ Since $t = -3.120 < -2.060 = t_{25;\,0.025}$, we reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$; $0.383 < \mu_2 - \mu_1 < 1.871$. }

(*b*) The I.Q.'s of 16 students from one area of a city showed a mean of 107 with a standard deviation of 10, while the I.Q.'s of 14 students from another area of the city showed a mean of 115 with a standard deviation of 8. Is there a significant difference between the I.Q.'s of the two groups at (*i*) 0.01 and (*ii*) 0.05 level of significance?

{ (*i*) Since $t_{28;\,0.005} = -2.763 < t = -2.395 < 2.763 = t_{28;\,0.995}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$ at $\alpha = 0.01$.

(*ii*) Since $t = -2.395 < -2.048 = t_{28;\,0.025}$, we reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$; at $\alpha = 0.05$. }

9. (*a*) Means of random samples, each of size 10, from two normal populations with the same standard deviation were found to be 16 and 20 respectively. Further, the sample standard deviations were equal to 5 and 7 respectively. Test the hypotheses that the populations have the same mean, using 0.05 level of significance.

{ Since $t_{18;\,0.025} = -2.101 < t = -1.47 < 2.101 = t_{18;\,0.975}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. }

(*b*) A random sample of size $n_1 = 10$, selected from a normal population has a mean $\bar{x}_1 = 20$ and a standard deviation $\hat{s}_1 = 5$. A second random sample of size $n_2 = 12$, selected from a different normal population has a mean $\bar{x}_2 = 24$ and a standard deviation $\hat{s}_2 = 6$. If $\mu_1 = 22$ and $\mu_2 = 19$ and σ_1^2 and σ_2^2 are unknown but

approximately equal, test whether there is any reason to doubt that $\mu_1 - \mu_2 = 3$.

{ Since $t = -2.934 < -2.086 = t_{20;\,0.025}$, we reject $H_0: \mu_1 - \mu_2 = 3$ in favour of $H_1: \mu_1 - \mu_2 \neq 3$ at $\alpha = 0.05$. }

10. (*a*) The following values are obtained by two different samples while sampling a normal population with $\mu = 3.0$ and $\sigma = 0.5$. Using these data solve the following problems:

A	2.7	3.9	2.6	2.8	3.2	3.6	2.7	3.0	3.8	3.1
B	2.4	2.8	3.0	3.3	3.7	3.3	3.1	2.7	2.4	3.3

(*i*) Combine the data into one set and test the hypothesis $H_0: \mu = 3.0$ against $H_1: \mu \neq 3.0$.

(*ii*) Assuming that σ is unknown test the hypothesis $H_0: \mu_A \leq \mu_B$ against $H_1: \mu_A > \mu_B$. Use $\alpha = 0.05$ in both cases.

{ (*i*) Since $z_{0.025} = -1.96 < z = 0.626 < 1.96 = z_{0.975}$, we do not reject $H_0: \mu = 3$ against $H_1: \mu_1 \neq 3$. (*ii*) Since $t = 0.694 < 1.734 = t_{18;\,0.95}$, we do not reject $H_0: \mu_A \leq \mu_B$ against $H_1: \mu_A > \mu_B$. }

(*b*) A group of 12 students are found to have the following I.Q.'s:

112, 109, 125, 113, 116, 131, 112, 123, 108, 113, 132, 128

Is it reasonable to assume that these students have come from a large population whose mean I.Q.'s is 115?

Another group of 10 students are found to have the following I.Q.'s:

117, 110, 106, 109, 116, 119, 107, 106, 105, 108

Can we conclude that both the groups of students have come from the same population?

{ (*i*) Since $t_{11;\,0.025} = -2.201 < t = 1.386 < 2.201 = t_{11;\,0.975}$, we do not reject $H_0: \mu = 115$ against $H_1: \mu \neq 115$ at $\alpha = 0.05$.

(*ii*) Since $t = 2.608 > 2.086 = t_{20;\,0.975}$, we reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$; at $\alpha = 0.05$. }

(*c*) A random sample of 16 values from a normal population gave a mean of 42 inches and a sum of squared deviations from this mean as 135 (inches)^2. Test the hypothesis that the mean in the population is 43.5 inches.

Another random sample of 9 values from another normal population gave a mean of 41.5 inches and a sum of squares of deviations from this mean as 128 (inches)^2. Test the hypothesis that mean of first population equals the mean of the second population, assuming that the variances of the two populations are equal.

{ Since $t_{15;\,0.025} = -2.131 < t = -2 < 2.131 = t_{15;\,0.975}$, we do not reject $H_0: \mu = 43.5$ against $H_1: \mu \neq 43.5$.)

Since $t_{23;\,0.025} = -2.069 < t = 0.355 < 2.069 = t_{23;\,0.975}$, we do not reject $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$ }

13.5 INFERENCES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION MEANS—*DEPENDENT SAMPLES*

There are many situations that require matched-pair comparisons. This method is appropriate when the observations of two dependent normal populations are compared. For example, if we are making inferences about the difference between blood pressure before and after administering a drug to heart patients, the blood pressure data after the drug is taken is dependent of the blood pressure before the drug is taken.

When matched samples are employed for making inference about the difference between two population means $\mu_2 - \mu_1$, it turns out that the procedures simplify to those for a single population mean μ. Suppose that we have an extreme and simplified form paired observations with pairing of before and after measurements. To analyse paired experiments, we consider for each pair the measurement before and after any conditions as variables X and Y, respectively.

Suppose that X is a normal random variable with mean μ_1 and variance σ_1^2, *i. e.*, $X \sim N(\mu_1, \sigma_1^2)$; Y is a normal random variable with mean μ_2 and variance σ_2^2, *i. e.*, $Y \sim N(\mu_2, \sigma_2^2)$; X and Y are dependent. We wish to estimate $\mu_2 - \mu_1$.

We consider for each pair the difference between the random variables X and Y and denote this random variable by D.

$$D = Y - X$$

We now consider the population of differences so obtained. We denote the mean of this population of differences by μ_D and variance by σ_D^2, which are given by

$$\mu_D = \mu_2 - \mu_1$$

$$\sigma_D^2 = \sigma_2^2 + \sigma_1^2 - 2\rho\,\sigma_1\,\sigma_2$$

A simple random sample of size n is selected from this population of differences. Let X_i and Y_i denote the before and after measurements respectively, for the *i-th* object in the random sample, so (X_i, Y_i) is a matched pair of observations.

Thus (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n), is a random sample of n paired observations from the bivariate normal distribution with parameters given by $\mu_1 = E(X)$, $\mu_2 = E(Y)$, $\sigma_1^2 = Var(X)$, $\sigma_2^2 = Var(Y)$, and $\rho = Corr(X, Y) = Cov(X, Y)/(\sigma_1\sigma_2)$.

The object is to make inferences about $\mu_D = \mu_2 - \mu_1$. To measure the change in measurements of the *i-th* object, we use the difference

$$D_i = Y_i - X_i, \qquad i = 1, 2, \cdots, n$$

then $D_1, D_2, \cdots, D_n$ are independently and identically distributed random variables with common normal distribution having mean μ_D and variance σ_D^2.

Now these differences $D_i = Y_i - X_i$, $i = 1, 2, \cdots, n$ may be thought of as a random sample of differences from a population of differences. We calculate the mean of the sample differences $D_1, D_2, \cdots, D_n$, denoted by $\overline{D}$. This statistic $\overline{D}$ is a random variable that has a sampling distribution with μ_D by the earlier procedures for a single population mean, but we are really estimating $\mu_D = \mu_2 - \mu_1$, the difference between the means of the two populations. The random variable

$$T = \frac{\overline{D} - \mu_D}{\hat{S}_D/\sqrt{n}}$$

follows a t-distribution with $\nu = n - 1$ degrees of freedom, where

$$\overline{D} = \frac{\sum_{i=1}^{n} D_i}{n}$$

$$\hat{S}_D^2 = \frac{\sum_{i=1}^{n}(D_i - \overline{D})^2}{n-1} = \frac{\sum_{i=1}^{n} D_i^2 - n\overline{D}^2}{n-1}$$

$$= \frac{\sum_{i=1}^{n} D_i^2 - \left(\sum_{i=1}^{n} D_i\right)^2 \Big/ n}{n-1}$$

$$= \frac{n\sum_{i=1}^{n} D_i^2 - \left(\sum_{i=1}^{n} D_i\right)^2}{n(n-1)}$$

13.5.1 Test of Hypothesis about $\mu_D = \mu_2 - \mu_1$.

Like the construction of confidence intervals for $\mu_D = \mu_2 - \mu_1$, tests on $\mu_D = \mu_2 - \mu_1$ for the case of dependent (matched) samples parallel those for single population. Mean of the sample differences $\overline{D}$ correspond to $\overline{X}$, mean of population of differences $\mu_D = \mu_2 - \mu_1$ corresponds to μ and σ_D corresponds σ. Small sample test of hypothesis $H_0: \mu_D = \delta_0$ is based on the test statistic

$$T = \frac{\overline{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$$

which has a t-distribution with $\nu = n - 1$ degrees of freedom.

The summary of the procedure of testing a hypothesis about the difference between means of two normal populations with matched pairs of observations, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing difference between means of two normal populations, Dependent samples: Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \mu_D \geq \delta_0$ $H_1: \mu_D < \delta_0$	$H_0: \mu_D \leq \delta_0$ $H_1: \mu_D > \delta_0$	$H_0: \mu_D = \delta_0$ $H_1: \mu_D \neq \delta_0$

Set level of significance: α

$$T = \frac{\overline{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$$

$$\overline{D} = \frac{\sum D_i}{n} \qquad \hat{S}_D = \sqrt{\frac{\sum (D_i - \overline{D})^2}{n-1}}$$

Reject H_0 if $T < t_{v;\alpha}$ $v = n-1$	Reject H_0 if $T > t_{v;1-\alpha}$	Reject H_0 if $T < t_{v;\alpha/2}$ or $T > t_{v;1-\alpha/2}$

Calculate the observed value of test statistic

$$t = \frac{\bar{d} - \delta_0}{\hat{s}_D/\sqrt{n}}$$

$$\bar{d} = \frac{\sum d_i}{n} \qquad \hat{s}_D = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}}$$

We reject H_0 if $t < t_{v;\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t > t_{v;1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $t < t_{v;\alpha/2}$ or $t > t_{v;1-\alpha/2}$ otherwise we do not reject H_0

Assumptions. To make valid inferences about the difference between means of two normal populations with matched pair of observations (dependent samples) the following assumptions must be met.

(*i*) A random sample of n differences must be selected from the population of differences

(*ii*) The population of differences must be (approximately) normally distributed.

The assumptions of an underlying normal distribution can be relaxed when the sample size is large. Applying the Central Limit Theorem to the differences $D_1, D_2, \cdots, D_n$ suggests a nearly normal distribution of $(\overline{D} - \mu_D)/(\hat{S}_D/\sqrt{n})$ when n is large (say greater than 30).

Example 13.24 *A new weight reducing technique, consisting of a liquid protein diet, is currently under going tests by the Food and Drug Administration (FDA) before its introduction into the market. A typical test performed by the FDA is the following. The weights of a random sample of 5 people are recorded before they are introduced to the liquid protein diet. The five individuals are then introduced to follow the liquid protein diet for 4 weeks. At the end of this period, their weights (in pounds) are again recorded. The results are listed in the table.*

Person	1	2	3	4	5
Weight before	150	195	188	197	204
Weight after	143	190	185	191	200

Perform a test of hypothesis at the 5% level of significance if the mean weight is smaller after the diet is used than before the diet is used.

Solution. The mean and standard deviation of the sample differences are

Person i	Weight Before x_i	Weight After y_i	Difference (after minus before) $d_i = y_i - x_i$	d_i^2
1	150	143	−7	49
2	195	190	−5	25
3	188	185	−3	9
4	197	191	−6	36
5	204	200	−4	16
Sums			−25	135

$$\bar{d} = \frac{\sum d_i}{n} = \frac{-25}{5} = -5$$

$$\hat{s}_D = \sqrt{\frac{\sum d_i^2 - n\bar{d}^2}{n-1}} = \sqrt{\frac{135 - 5(-5)^2}{5-1}} = 1.58$$

The experimenter believes that the mean weight after is smaller than before, then the mean "after minus before" difference μ_D is less than zero. Hence $\mu_D < 0$ is the alternative hypothesis. The elements of the one-sided left tail test of hypothesis are:

Null hypothesis $H_0: \mu_D \geq 0$

Alternative hypothesis $H_1: \mu_D < 0$

Level of significance: $\alpha = 0.05$

Test statistic: $T = \dfrac{\overline{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $v = n - 1 = 5 - 1 = 4$

Critical value: $t_{v;\alpha} = t_{4;0.05} = -2.132$ (From Table 12)

Critical region: $T < -2.132$

Decision rule: Reject H_0 if $T < -2.132$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\overline{d} - \delta_0}{\hat{s}_D/\sqrt{n}} = \dfrac{-5-0}{1.58/\sqrt{5}} = -7.076$

Conclusion: Since $t = -7.076 < -2.132$, so we reject H_0.

Example 13.25 *Productivity (units produced per day) for a random sample of 10 workers was recorded before and after training. The following paired observations were obtained.*

Worker	1	2	3	4	5	6	7	8	9	10
Productivity before	54	56	50	52	55	52	56	53	53	60
Productivity after	60	59	57	56	56	58	62	55	54	64

Perform a test of hypothesis at the 1 percent level of significance to determine if mean productivity is greater after training than before training.

Solution. The mean and standard deviation of the sample differences are

Worker	Units produced per day		Difference		
	Before	After	(after minus before)		
i	x_i	y_i	$d_i = y_i - x_i$	$d_i - \overline{d}$	$(d_i - \overline{d})^2$
1	54	60	6	2	4
2	56	59	3	−1	1
3	50	57	7	3	9
4	52	56	4	0	0
5	55	56	1	−3	9
6	52	58	6	2	4
7	56	62	6	2	4
8	53	55	2	−2	4
9	53	54	1	−3	9
10	60	64	4	0	0
Sums			$\sum d_i = 40$		$\sum(d_i - \overline{d})^2 = 44$

$$\overline{d} = \frac{\sum d_i}{n} = \frac{40}{10} = 4$$

$$\hat{s}_D = \sqrt{\frac{\sum(d_i - \overline{d})^2}{n-1}} = \sqrt{\frac{44}{10-1}} = 2.21$$

The experimenter believes that the mean productivity after is greater than before, then the population mean "after minus before" difference μ_D is greater than zero. Hence $\mu_D > 0$ is the alternative hypothesis. The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu_D \le 0$

Alternative hypothesis $H_1: \mu_D > 0$

Level of significance: $\alpha = 0.01 \Rightarrow 1 - \alpha = 0.99$

Test statistic: $T = \dfrac{\overline{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $v = n - 1 = 10 - 1 = 9$

Critical value: $t_{v;1-\alpha} = t_{9;0.99} = 2.821$ (From Table 12)

Critical region: $T > 2.821$

Decision rule: Reject H_0 if $T > 2.821$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\overline{d} - \delta_0}{\hat{s}_D/\sqrt{n}} = \dfrac{4-0}{2.21/\sqrt{10}} = 5.72$

Conclusion: Since $t = 5.72 > 2.821$, we reject H_0

Example 13.26 *A company is interested in hiring a new secretary. Several candidates are interviewed and the choice is narrowed to two possibilities. The final choice will be based on typing ability. Six letters are randomly selected from the company's file, and each candidate is required to type each one. The number of words typed per minute is recorded to each candidate. The data are listed in the following table.*

Letter	1	2	3	4	5	6
Candidate A	62	60	65	58	59	64
Candidate B	59	60	61	57	55	60

Do the date provide sufficient evidence to indicate a difference in the mean number of words typed per minute by the two candidates. Test using $\alpha = 0.02$.

Solution. The mean and standard deviation of the sample differences are

Letter	Number of words typed by		Difference	
	Candidate A	Candidate B	(A minus B)	
i	x_i	y_i	$d_i = x_i - y_i$	d_i^2
1	62	59	3	9
2	60	60	0	0
3	65	61	4	16
4	58	57	1	1
5	59	55	4	16
6	64	60	4	16
Sums			16	58

$$\bar{d} = \frac{\Sigma d_i}{n} = \frac{16}{6} = 2.667$$

$$\hat{s}_D = \sqrt{\frac{n\Sigma d_i^2 - (\Sigma d_i)^2}{n(n-1)}} = \sqrt{\frac{6(58) - (16)^2}{6(6-1)}} = 1.751$$

The experimenter believes that the mean typing rate differs for the candidate A and B, the population mean "A minus B" difference μ_D is not equal to zero. Hence $\mu_D \neq 0$ is the alternative hypothesis. The elements of the two-sided test of hypothesis are:

Null hypothesis $H_0: \mu_D = 0$

Alternative hypothesis $H_1: \mu_D \neq 0$

Level of significance: $\alpha = 0.02 \Rightarrow \alpha/2 = 0.01 \Rightarrow 1 - \alpha/2 = 0.99$

Test statistic: $T = \dfrac{\bar{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n - 1 = 6 - 1 = 5$

Critical values: $t_{\nu;\,\alpha/2} = t_{5;\,0.01} = -3.365$,

$t_{\nu;\,1-\alpha/2} = t_{5;\,0.99} = 3.365$ (From Table 12)

Critical region: $T < -3.365$ or $T > 3.365$

Decision rule: Reject H_0 if $T < -3.365$ or $T > 3.365$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\bar{d} - \delta_0}{\hat{s}_D/\sqrt{n}} = \dfrac{2.667 - 0}{1.751/\sqrt{6}} = 3.731$

Conclusion: Since $t = 3.731 > 3.365$, so we reject H_0.

Example 13.27 *An experiment was performed with seven hop plants. One half of each plant was pollinated and the other half was not pollinated. The yield of the seed of each hop plant is tabulated as follows:*

Plant	*Pollinated*	*Non-pollinated*
1	0.78	0.21
2	0.76	0.12
3	0.43	0.32
4	0.92	0.29
5	0.86	0.30
6	0.59	0.20
7	0.68	0.14

Determine at the 5% level whether the pollinated half of the plant gives a higher yield in seed than the non-pollinated half. State the assumptions and hypothesis to be tested and carry through the computations to make a decision

Solution. The mean and standard deviation of the sample differences are

Plant	Yield in seed		Difference	
	Pollinated	Non-pollinated	(Pollinated minus Non-pollinated)	
i	x_i	y_i	$d_i = x_i - y_i$	d_i^2
1	0.78	0.21	0.57	0.3249
2	0.76	0.12	0.64	0.4096
3	0.43	0.32	0.11	0.0121
4	0.92	0.29	0.63	0.3969
5	0.86	0.30	0.56	0.3136
6	0.59	0.20	0.39	0.1521
7	0.68	0.14	0.54	0.2916
Sums			3.44	1.9008

$$\bar{d} = \frac{\Sigma d_i}{n} = \frac{3.44}{7} = 0.491$$

$$\hat{s}_D = \sqrt{\frac{n\Sigma d_i^2 - (\Sigma d_i)^2}{n(n-1)}} = \sqrt{\frac{7(1.9008) - (3.44)^2}{7(7-1)}} = 0.1872$$

The experimenter believes that the pollinated half gives a higher mean yield in seed, then the alternative hypothesis is $\mu_D > 0$. The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \mu_D \leq 0$

Alternative hypothesis $H_1: \mu_D > 0$

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $T = \dfrac{\bar{D} - \delta_0}{\hat{S}_D/\sqrt{n}}$ follows a t-distribution under H_0 with

Degrees of freedom: $\nu = n - 1 = 7 - 1 = 6$

Critical values: $t_{\nu;\,1-\alpha} = t_{6;\,0.95} = 1.943$ (From Table 12)

Critical region: $T > 1.943$

Decision rule: Reject H_0 if $T > 1.943$, otherwise do not reject H_0.

Observed value: $t = \dfrac{\bar{d} - \delta_0}{\hat{s}_D/\sqrt{n}} = \dfrac{0.491 - 0}{0.1872/\sqrt{7}} = 6.939$

Conclusion: Since $t = 6.939 > 1.943$, so we reject H_0, and conclude that the data provides a sufficient evidence that pollination gives a higher mean yield in seed.

Exercise 13.5

1. (a) Haemoglobin values were determined on six patients before starting and after three weeks on B_{12} Therapy. The following data were obtained

Individual Number	Haemoglobin (gm) Before Therapy	Haemoglobin (gm) After Therapy
1	12.2	13.0
2	11.3	13.4
3	14.7	16.0
4	11.4	13.6
5	11.5	14.0
6	12.7	13.8

Do the data indicate a significant improvement ?
(Since $t = 5.927 > 2.015 = t_{5;0.95}$, we reject $H_0: \mu_D \le 0$ in favour of $H_1: \mu_D > 0$)

(b) Eleven school boys were given a test in Drawing. They were given one month's further tuition and a second test of equal difficulty was held at the end of it.

Marks in 1st test	23	20	19	21	18	20	18	17	23	16	19
Marks in 2nd test	24	19	22	18	20	22	20	20	20	20	17

Do the marks give evidence that the students have benefited by the extra coaching.
(Since $t = 0.956 < 1.812 = t_{10;0.95}$, so we do not reject $H_0: \mu_D \le 0$ against $H_1: \mu_D > 0$)

2. (a) A taxi company is trying to decide whether the use of radial tires instead of regular belted tires improves fuel economy. Twelve cars were equipped with radial tires and driven over a prescribed test course. Without changing drivers, the same cars were then equipped with regular belted tires and driven once again over the test course. The gasoline consumption in km per litre, was recorded as follows:

Radial tires	4.2	4.7	6.6	7.0	6.7	4.5	5.7	6.0	7.4	4.9	6.1	5.2
Belted tires	4.1	4.9	6.2	6.9	6.8	4.4	5.7	5.8	6.9	4.7	6.0	4.9

At the 0.025 level of significance, can we conclude, that cars equipped with radial tires give better fuel economy than those equipped with belted tires ? Assume the population to be normally distributed.
(Since $t = 2.490 > 2.201 = t_{11;0.975}$, we reject $H_0: \mu_D \le 0$ in favour of $H_1: \mu_D > 0$)

(b) A certain stimulus administered to each of the nine patients resulted in the following increase in blood pressure:

5, 1, 8, 0, 3, 3, 5, −2, 4

Can it be concluded that blood pressure in general is increased by administering the stimulus.
(Since $t = 3.0 > 1.86 = t_{8;0.95}$, we reject $H_0: \mu_D \le 0$ in favour of $H_1: \mu_D > 0$)

3. (a) To verify whether a course in Statistics improved performance a similar test was given to 12 participants both before and after the course. The original grades recorded in alphabetical order of the participants were 44, 40, 61, 52, 32, 44, 70, 41, 67, 72, 53 and 72. After the course the grades were, in the same order, 53, 38, 69, 57, 46, 39, 73, 48, 73, 74, 60 and 78.

(i) Was the course useful, as measured by performance on the test ? Consider these 12 participants as a sample from a population.

(ii) Would the same conclusion be reached if the tests were not considered paired ? (use 5% level of significance in both cases).

[(i) Since $t = 3.445 > 1.796 = t_{11;0.95}$, we reject $H_0: \mu_D \le 0$ in favour of $H_1: \mu_D > 0$.

(ii) Since $t = 0.863 < 1.717 = t_{22;0.95}$, we do not reject $H_0: \mu_2 - \mu_1 \le 0$ against $H_1: \mu_2 - \mu_1 > 0$].

(b) The time required by 10 persons to perform a task in seconds before and after receiving a mild stimulant are given in the accompanying table.

Before	34	45	31	43	40	41	33	29	41	37
After	29	42	32	29	36	42	26	28	38	33

Test the hypothesis that there is no difference between the mean times in the "before" and "after" populations. As an alternative, assumed that the after population will have a lower mean. Use 5% level of significance.
(Since $t = -2.83 < -1.833 = t_{9;0.05}$, we reject $H_0: \mu_D \ge 0$ in favour of $H_1: \mu_D < 0$.)

4. (a) Ten young recruits were put through a physical training programme by the army. Their weights were recorded before and after the training with the following results:

Recruit	1	2	3	4	5	6	7	8	9	10
Weight (Before)	127	126	162	170	143	205	168	175	197	136
Weight (After)	135	200	160	182	147	200	172	186	193	141

Using $\alpha = 0.05$, should we conclude that the training programme affects the average weight of young recruits.
(Since $t_{9;0.025} = -2.262 < t = 1.471 < 2.262 = t_{9;0.975}$, we do not reject $H_0: \mu_D = 0$ against $H_1: \mu_D \ne 0$.)

(b) The following data give paired yields of two varieties of wheat. Each pair was planted in a different locality.

Variety I	45	32	58	57	60	38	47	51	42	38
Variety II	47	34	60	59	63	44	49	53	46	41

Test the hypothesis that the mean yields are equal.
(Since $t = 6.725 > 2.262 = t_{9;\,0.975}$, we reject $H_0: \mu_D = 0$ in favour of $H_1: \mu_D \neq 0$.)

5. (a) Twenty college freshmen were divided into 10 pairs, each member of the pair having approximately the same I.Q. One of each pair was selected at random and assigned to a Mathematics section using programmed materials only. The other members of each pair were assigned to a section in which the teacher lectured. At the end of the semester each group was given the same examination and the following results were recorded:

Pair	Programmed materials	Lectures
1	76	81
2	70	52
3	85	87
4	58	70
5	91	86
6	75	77
7	82	90
8	64	73
9	79	85
10	88	83

Test the hypothesis that there is no difference between the mean scores in the "programmed material" and "lectures" populations. Use 5% level of significance.
(Since $t_{9;\,0.025} = -2.262 < t = 0.571 < 2.262 = t_{9;\,0.975}$, we do not reject $H_0: \mu_D = 0$ against $H_1: \mu_D \neq 0$.)

(b) It is claimed that a new diet will reduce a person's weight by at least 10 pounds on the average in a period of 2 weeks. The weight of 7 women who followed this diet were recorded before and after a 2-week period.

Woman	1	2	3	4	5	6	7
Weight (Before)	129	133	136	152	141	138	125
Weight (After)	130	121	128	137	129	132	120

Test the manufacturer's claim at a 5% level of significance for the mean difference in weights. Assume the distribution of weights before and after to be approximately normal.
(Since $t = 0.910 < 1.943 = t_{6;\,0.95}$, we do not reject $H_0: \mu_D \leq -10$ against $H_1: \mu_D > -10$)

13.6 TEST OF HYPOTHESIS ABOUT THE DIFFERENCE BETWEEN TWO POPULATION PROPORTIONS, $\pi_1 - \pi_2$

There are many economic and management problems where we must decide whether observed differences between two sample proportions come from common or different populations. Such problems require a comparison between the rates of incidence of a characteristic in two populations.

13.6.1 Forms of Hypothesis. We are interested in tests about the parameter $\pi_1 - \pi_2$. Let δ_0 be the hypothesized value of the difference between two population proportions, then the three possible null hypotheses about the difference between population proportions, and their corresponding alternative hypotheses, are:

1. $H_0: \pi_1 - \pi_2 \geq \delta_0$ against $H_1: \pi_1 - \pi_2 < \delta_0$
2. $H_0: \pi_1 - \pi_2 \leq \delta_0$ against $H_1: \pi_1 - \pi_2 > \delta_0$
3. $H_0: \pi_1 - \pi_2 = \delta_0$ against $H_1: \pi_1 - \pi_2 \neq \delta_0$

Depending upon the alternative hypothesis a one-sided left tail test is required for (1); (2) requires a one-sided right tail test, and (3) requires a two-tail test.

13.6.2 Test based on Normal Distributions. The unknown proportion of elements possessing the particular characteristic in population I and in population II are denoted by π_1 and π_2, respectively. A random sample of size n_1 is taken from population I and the number of successes is denoted by X_1. An independent random sample of size n_2 is taken from population II and the number of successes is denoted by X_2. The sample proportions are:

$$P_1 = \frac{X_1}{n_1} \quad \text{and} \quad P_2 = \frac{X_2}{n_2}$$

An intuitively appealing estimator for $\pi_1 - \pi_2$ is the difference between the sample proportions $P_1 - P_2$. When testing hypothesis about $\pi_1 - \pi_2$, we will use the sampling distribution of $P_1 - P_2$. The sampling distribution of $P_1 - P_2$ will have mean and standard error as

$$\mu_{P_1 - P_2} = \pi_1 - \pi_2, \qquad \sigma_{P_1 - P_2} = \sqrt{\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}}$$

For large sample sizes n_1 and n_2, the random variable

$$Z = \frac{(P_1 - P_2) - (\pi_1 - \pi_2)}{\sqrt{\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}}}$$

is approximately standard normal. The estimate of the standard error of $P_1 - P_2$ can be obtained by replacing π_1 and π_2 by their sample estimates P_1 and P_2 as

$$\hat{\sigma}_{P_1 - P_2} = \sqrt{\frac{P_1(1-P_1)}{n_1} + \frac{P_2(1-P_2)}{n_2}}$$

The random variable Z then becomes

$$Z = \frac{(P_1 - P_2) - (\pi_1 - \pi_2)}{\sqrt{\dfrac{P_1(1-P_1)}{n_1} + \dfrac{P_2(1-P_2)}{n_2}}}$$

(a) ***Testing the Hypothesis that the Difference between Two Population Proportions Equals Some Non-zero Value.*** To test the more general hypothesis $H_0: \pi_1 - \pi_2 = \delta_0$, we use the test statistic

$$Z = \frac{(P_1 - P_2) - \delta_0}{\sqrt{\dfrac{P_1(1-P_1)}{n_1} + \dfrac{P_2(1-P_2)}{n_2}}}$$

(b) ***Testing the Hypothesis that Two Independent Populations have Same Proportion of Successes.*** Our aim is to test the null hypothesis of no difference $H_0: \pi_1 = \pi_2$. Under the null hypothesis $H_0: \pi_1 - \pi_2 = 0$, the two populations have equal proportions $\pi_1 = \pi_2$, we denote the unspecified common population proportion by π.

Since under the null hypothesis it is assumed that $\pi_1 = \pi_2 = \pi$, then the sampling distribution of $P_1 - P_2$ will have mean

$$\mu_{P_1 - P_2} = \pi - \pi = 0$$

and standard error

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi(1-\pi)}{n_1} + \frac{\pi(1-\pi)}{n_2}} = \sqrt{\pi(1-\pi)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

The unknown population proportion π involved in the standard error must now be replaced by the sample proportion.

The difficulty in making this replacement lies in the fact that we have two estimates of π, P_1 and P_2, since two different samples were collected. Which of these two estimates should be used to estimate the unknown population proportion π.

Since we wish to obtain the best estimate available, it would seem reasonable to use an estimator that would pool the information from the two samples.

The proportion of successes in the combined sample provides the pooled estimate.

$$\hat{\pi} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2}$$

The estimate of standard error then becomes

$$\hat{\sigma}_{P_1 - P_2} = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

For large sample sizes n_1 and n_2, the random variable

$$Z = \frac{P_1 - P_2}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

is approximately standard normal under the null hypothesis $H_0: \pi_1 - \pi_2 = 0$.

The summary of the procedure of testing a hypothesis that the difference between two population proportions equals some none-zero value, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing that the Difference between Two Population Proportions Equals Some None-zero Value. Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \pi_1 - \pi_2 \geq \delta_0$ $H_1: \pi_1 - \pi_2 < \delta_0$	$H_0: \pi_1 - \pi_2 \leq \delta_0$ $H_1: \pi_1 - \pi_2 > \delta_0$	$H_0: \pi_1 - \pi_2 = \delta_0$ $H_1: \pi_1 - \pi_2 \neq \delta_0$

Set level of significance: α

$$Z = \frac{(P_1 - P_2) - \delta_0}{\sqrt{\dfrac{P_1(1-P_1)}{n_1} + \dfrac{P_2(1-P_2)}{n_2}}}$$

Reject H_0 if $Z < z_{\alpha}$	Reject H_0 if $Z > z_{1-\alpha}$	Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

Calculate the observed value of test statistic

$$z = \frac{(p_1 - p_2) - \delta_0}{\sqrt{\dfrac{p_1(1-p_1)}{n_1} + \dfrac{p_2(1-p_2)}{n_2}}}$$

We reject H_0 if $z < z_{\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z > z_{1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ otherwise we do not reject H_0

The summary of the procedure of testing a hypothesis that two independent populations have same proportion of successes, at a given significance level α, for the three respective alternative hypotheses is shown in the following table.

Testing that Two Independent Populations have Same Proportion of Successes: Summary of procedure for 3 alternatives

Choose alternative hypothesis

$H_0: \pi_1 - \pi_2 \geq 0$ $H_1: \pi_1 - \pi_2 < 0$	$H_0: \pi_1 - \pi_2 \leq 0$ $H_1: \pi_1 - \pi_2 > 0$	$H_0: \pi_1 - \pi_2 = 0$ $H_1: \pi_1 - \pi_2 \neq 0$

Set level of significance: α

$$Z = \frac{(P_1 - P_2) - 0}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$\hat{\pi} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2}$$

Reject H_0 if $Z < z_\alpha$	Reject H_0 if $Z > z_{1-\alpha}$	Reject H_0 if $Z < z_{\alpha/2}$ or $Z > z_{1-\alpha/2}$

Calculate the observed value of test statistic

$$z = \frac{(p_1 - p_2) - 0}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$\hat{\pi} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

We reject H_0 if $z < z_\alpha$ otherwise we do not reject H_0	We reject H_0 if $z > z_{1-\alpha}$ otherwise we do not reject H_0	We reject H_0 if $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$ otherwise we do not reject H_0

Example 13.28 *A cigarette manufacturing firm distributes two brands of cigarettes. It is found that 56 of 200 smokers prefer brand "A" and that 29 of 150 smokers prefer brand "B". Test at 0.06 level of significance that brand "A" outsell brand "B" by at least 10% against the alternative hypothesis that the difference is less than 10%.*

Solution. We have $n_1 = 200, \quad x_1 = 56, \quad p_1 = \frac{x_1}{n_1} = \frac{56}{200} = 0.28$

$n_2 = 150, \quad x_2 = 29, \quad p_2 = \frac{x_2}{n_2} = \frac{29}{150} = 0.193$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the difference between the two proportions of smokers $\pi_1 - \pi_2$ is less than 0.1. Consequently, we will test the null hypothesis that $\pi_1 - \pi_2 \geq 0.1$ against the alternative hypothesis that $\pi_1 - \pi_2 < 0.1$.

The elements of the one-sided left tail test of hypothesis are:

Null hypothesis $H_0: \pi_1 - \pi_2 \geq 0.1$ (*i. e.,* $\delta_0 \geq 0.1$)

Alternative hypothesis $H_1: \pi_1 - \pi_2 < 0.1$ (*i. e.,* $\delta_0 < 0.1$)

Level of significance: $\alpha = 0.06$

Test statistic:

$$Z = \frac{(P_1 - P_2) - \delta_0}{\sqrt{\frac{P_1(1-P_1)}{n_1} + \frac{P_2(1-P_2)}{n_2}}}$$

follows approximately standard normal distribution under H_0

Critical value: $z_\alpha = z_{0.06} = -1.555$ { From Table 10 (*a*) }

Critical region: $Z < -1.555$

Decision rule: Reject H_0 if $Z < -1.555$, otherwise do not reject H_0.

Observed value:

$$z = \frac{(p_1 - p_2) - \delta_0}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$

$$= \frac{(0.28 - 0.193) - 0.10}{\sqrt{\frac{0.28\,(1-0.28)}{200} + \frac{0.193\,(1-0.193)}{150}}} = -0.287$$

Conclusion: Since $z = -0.287 > -1.555 = z_{0.06}$, we do not reject $H_0: \pi_1 - \pi_2 \geq 0.10$ and conclude that the data present sufficient evidence to indicate that brand A outsells brand B by at least 10%.

Example 13.29 *A firm found with the help of a sample survey (size of sample 900) that 0.75 of the population consumes things produced by them. The firm then advertised the goods in paper* *and on radio. After one year, sample of size 1000 reveals that proportion of consumers of the goods produced by the firm is now 0.8. Is this rise significant indicating that the advertisement was effective.*

Solution. We have $n_1 = 900,\ p_1 = 0.75,\ n_2 = 1000,\ p_2 = 0.80$

$$\hat{\pi} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$= \frac{900(0.75) + 1000(0.80)}{900 + 1000} = 0.776$$

The objective of the sampling is to attempt to support the research (alternative) hypothesis that the proportion of consumers after advertisement π_2 is greater than that of before π_1. Consequently, we will test the null hypothesis that $\pi_2 \le \pi_1$ against the alternative hypothesis that $\pi_2 > \pi_1$.

The elements of the one-sided right tail test of hypothesis are:

Null hypothesis $H_0: \pi_2 \le \pi_1 \Rightarrow \pi_2 - \pi_1 \le 0$ (*i. e.,* $\delta_0 \le 0$)

Alternative hypothesis $H_1: \pi_2 > \pi_1 \Rightarrow \pi_2 - \pi_1 > 0$ (*i. e.,* $\delta_0 > 0$)

Level of significance: $\alpha = 0.05 \Rightarrow 1 - \alpha = 0.95$

Test statistic: $$Z = \frac{(P_2 - P_1) - 0}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$ follows approximately standard normal distribution under H_0

Critical value: $z_{1-\alpha} = z_{0.95} = 1.645$ { From Table 10 *(b)* }

Critical region: $Z > 1.645$

Decision rule: Reject H_0 if $Z > 1.645$, otherwise do not reject H_0.

Observed value: $$z = \frac{(p_2 - p_1) - 0}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(0.8 - 0.75) - 0}{\sqrt{0.776(1-0.776)\left(\frac{1}{900} + \frac{1}{1000}\right)}} = 2.61$$

Conclusion: Since $z = 2.61 > 1.645 = z_{0.95}$, we reject H_0 and conclude that the data present sufficient evidence to indicate the consumption after advertisement is higher than that of before.

Exercise 13.6

1. (a) For each of the following sets of data, test the hypothesis that there is a common proportion π.

	Sample I		Sample II			
	Sample size	Number of Successes	Sample size	Number of Successes	Hypotheses	Level of significance
(i)	150	125	200	176	$H_0: \pi_1 \ge \pi_2$ $H_1: \pi_1 < \pi_2$	5%
(ii)	1000	542	900	427	$H_0: \pi_1 = \pi_2$ $H_1: \pi_1 \ne \pi_2$	1%
(iii)	100	63	120	80	$H_0: \pi_1 = \pi_2$ $H_1: \pi_1 \ne \pi_2$	10%
(iv)	80	40	60	23	$H_0: \pi_1 \le \pi_2$ $H_1: \pi_1 > \pi_2$	5%

{ *(i)* Since $z = -1.245 > -1.645 = z_{0.05}$, we do not reject $H_0: \pi_1 \ge \pi_2$ against $H_1: \pi_1 < \pi_2$.

(ii) Since $z = 2.941 > 2.576 = z_{0.995}$, we reject $H_0: \pi_1 = \pi_2$ in favour of $H_1: \pi_1 \ne \pi_2$.

(iii) Since $z_{0.05} = -1.645 < z = -0.573 < 1.645 = z_{0.95}$, we do not reject $H_0: \pi_1 = \pi_2$ against $H_1: \pi_1 \ne \pi_2$.

(iv) Since $z = 1.373 < 1.645 = z_{0.95}$, we do not reject $H_0: \pi_1 \le \pi_2$ against $H_1: \pi_1 > \pi_2$. }

(b) In a population that have certain minor blood disorder, samples of 100 males and 100 females are taken. It is found that 31 males and 24 females have the blood disorder. Can we conclude at 0.01 level of significance that proportion of men who have blood disorder is greater than proportion of women.

{ Since $z = 1.109 < 2.326 = z_{0.99}$, we do not reject $H_0: \pi_1 \le \pi_2$ against $H_1: \pi_1 > \pi_2$. }

2. (a) The records of a hospital show that 52 men in a sample of 1000 men versus 23 women in a sample of 1000 women were admitted because of heart disease. Do these data present sufficient evidence to indicate a higher rate of heart disease among men admitted to the hospital?

{ Since $z = 3.413 > 1.645 = z_{0.95}$, we reject $H_0: \pi_1 \le \pi_2$ in favour of $H_1: \pi_1 > \pi_2$. }

(*b*) In a study to estimate the proportion of housewives who own an automatic dryer, it is found that 63 of 100 urban residents have a dryer and 59 of 125 suburban residents own a dryer. Is there a significant difference between the proportions of urban and suburban housewives who own an automatic dryer? (use a 0.04 level of significance).
{ Since $z = 2.364 > 2.0537 = z_{0.98}$, so we reject $H_0: \pi_1 = \pi_2$ in favour of $H_1: \pi_1 \neq \pi_2$. }

3. (*a*) A random sample of 150 light bulbs manufactured by a firm *A* showed 12 defective bulbs while a random sample of 100 light bulbs manufactured by another firm *B* showed 4 defective bulbs. Is there a significant difference between the proportions defectives of the two firms?
{ Since $z_{0.025} = -1.960 < z = 1.270 < 1.960 = z_{0.975}$, we do not reject $H_0: \pi_1 = \pi_2$ against $H_1: \pi_1 \neq \pi_2$. }

(*b*) In a random sample of 800 adults from the population of a large city 600 are found to be smokers. In a random sample of 1000 adults from another large city, 700 are smokers. Do the data indicate that the cities are significantly different with respect to the prevalence of smoking among men.
{ Since $z = 2.353 > 1.960 = z_{0.975}$, we reject $H_0: \pi_1 = \pi_2$ in favour of $H_1: \pi_1 \neq \pi_2$. }

(*c*) In two large populations, there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in a sample of 1200 and 900 respectively from the two populations?
{ Since $z = 2.529 > 1.960 = z_{0.975}$, we reject $H_0: \pi_1 = \pi_2$ in favour of $H_1: \pi_1 \neq \pi_2$ at $\alpha = 0.05$. }

4. (*a*) A machine puts out 16 imperfect articles in a sample of 500. After the machine is overhauled, it puts out 3 imperfect articles in a batch of 100. Has the machine been improved? Use 5% level of significance.
(Since $z = -0.104 > -1.645 = z_{0.05}$, we do not reject $H_0: \pi_2 \geq \pi_1$ against $H_1: \pi_2 < \pi_1$)

(*b*) In a random sample of 250 persons who skipped breakfast 102 reported that they experienced mid morning fatigue, and in a random sample of 250 persons who ate breakfast, 73 reported that they experienced mid morning fatigue. Use 0.01 level of significance to test the null hypothesis that there is no difference between the corresponding proportions against the alternative hypothesis that the mid morning fatigue is more prevalent among persons who skip breakfast.
(Since $z = 2.719 > 2.326 = z_{0.99}$, we reject $H_0: \pi_1 \leq \pi_2$ in favour of $H_1: \pi_1 > \pi_2$)

5. (*a*) A manufacturer of house-dresses sent out advertising by mail. He sent samples of material to each of two groups of 1000 women. For one group he enclosed a white return envelope and for the other group, a blue envelope. He received orders from 9% and 12% respectively. Is it quite certain that the blue envelope will help sales.

Use $\alpha = 0.05$.
(Since $z = 2.19 > 1.645 = z_{0.95}$, we reject $H_0: \pi_2 \leq \pi_1$ in favour of $H_1: \pi_2 > \pi_1$)

(*b*) A random sample of 150 high school students was asked whether they would turn to their father or their mother for help with a homework assignment in Mathematics and another random sample of 150 high school students was asked the same question with regard to a homework assignment in English. Use the result shown in the following table and the 0.01 level of significance to test whether or not there is a difference between the true proportions of high school students who turn to their fathers rather than their mothers for help in these two subjects:

	Mathematics	*English*
Mother:	59	85
Father:	91	65

(Since $z = 3.016 > 2.576 = z_{0.995}$, we reject $H_0: \pi_1 = \pi_2$ in favour of $H_1: \pi_1 \neq \pi_2$)

Exercise 13.7

Objective Questions

1. Fill in the blanks.

(*i*) A statistical ——— is an assertion about the distribution of one or more random variables. (*hypothesis*)

(*ii*) A statistical hypothesis ——— is a procedure to determine whether or not an assumption about some parameter of population is supported by an observed random sample. (*testing*)

(*iii*) A ——— hypothesis is that hypothesis which is tested for possible rejection under the assumption that it is true. (*null*)

(*iv*) An ——— hypothesis is that hypothesis which we are willing to accept when the null hypothesis is rejected. (*alternative*)

(*v*) The ——— hypothesis always contains some form of an equality sign. (*null*)

(*vi*) The ——— hypothesis never contains the sign of equality and is always in an inequality from. (*alternative*) (*vii*) A statistic on the basis of which a decision is made about the hypotheses of interest is called ————. (*test statistic*)

(*viii*) A ———— region specifies a set of values of the test statistic for which the H_0 is rejected. (*rejection*)

(*ix*) An ———— region specifies a set of values of the test statistic for which the H_0 is not rejected. (*accepting*)

(*x*) The values of the test statistic which separate the rejection region from acceptance region are called ———— values. (*critical*)

(*xi*) If the critical region is located equally in both tails of the sampling distribution of test statistic, the test is called ———— test. (*two-tailed*)

(*xii*) If the critical region is located in only one tail of the sampling distribution of test statistic, the test is called ———— test. (*one-tailed*)

2. Fill in the blanks.

(*i*) A value of the test statistic is said to be statistically ———— if it falls in the acceptance region. (*insignificant*)

(*ii*) A value of the test-statistic is said to be statistically significant if it falls in the ———— region. (*critical*)

(*iii*) If the null hypothesis is false, we may accept it leading to a ———— decision. (*wrong*)

(*iv*) If the null hypothesis is false, we may reject it leading to a ———— decision. (*correct*)

(*v*) A ———— error is made by rejecting H_0 if H_0 is actually true. (*Type-I*)

(*vi*) A ———— error is made by accepting H_0 if H_1 is actually true. (*Type-II*)

(*vii*) The level of ———— of a test is the maximum probability with which we are willing to a risk of Type-I error. (*significance*)

(*viii*) The level of ———— is the probability of accepting a true null hypothesis. (*confidence*)

(*ix*) The ———— of freedom is the number of independent or freely chosen variables. (*degrees*)

3. Mark off the following statements as True or False.

(*i*) The types of statistical inferences are estimation of parameters and testing of hypotheses. (*true*)

(*ii*) A null hypothesis is rejected when a test statistic has a value that is not consistent with the null hypothesis. (*true*)

(*iii*) The probability of accepting the true null hypothesis is called the level of significance. (*false*)

(*iv*) The probability of rejecting the true null hypothesis is called the level of confidence. (*false*)

(*v*) The probability of accepting the true null hypothesis is called the level of confidence. (*true*)

(*vi*) An assumption made about the population parameter which may or may not be true is called statistical hypothesis. (*true*)

(*vii*) The null hypothesis and alternative hypothesis are complementary to each other. (*true*)

(*viii*) The null hypothesis H_0 always contain some from of an equality sign. (*true*)

(*ix*) The alternative H_1 never contains the sign of equality. (*true*)

4. Mark off the following statements as True or False.

(*i*) The level of significance is the probability of accepting a null hypothesis when it is true. (*false*)

(*ii*) A null hypothesis is rejected if the value of test-statistic is consistent with the H_0. (*false*)

(*iii*) The Type-I error is considered more serious than a Type-II error. (*true*)

(*iv*) A value of the test-statistic is said to be statistically insignificant if it falls in the rejection region. (*false*)

(v) A value of the test statistic is said to be statistically significant if it falls in the rejection region. (true)

(vi) If the critical region is located in only one tail of the sampling distribution of the test-statistic, then it is called one tailed or one-sided test. (true)

(vii) If the critical region is equally located in both tails of the sampling distribution of the test-statistic, then it is called a two-tailed or two sided test. (true)

(viii) The degrees of freedom is the number of dependent variables. (false)

(ix) The t-distribution approaches the normal distribution as the sample size increases. (true)

(x) The standardized normal distribution has smaller dispersion than student's t-distribution. (true)

(xi) H_0 is rejected when probability of its occurrence is equal to or less than level of significance. (true)

14 SIMPLE LINEAR REGRESSION AND CORRELATION

14.1 RELATIONS BETWEEN VARIABLES

The concept of a relation between two variables such as family incomes and family expenditures for housing, is a familiar one. We now distinguish between a *functional relation* and a *statistical relation*, and consider each of them in turn.

14.1.1 Functional Relation between Two Variables. A *functional relation* between two variables, is a perfect relation, where the value of the dependent variable is uniquely determined from the value of the independent variable. A functional relation is expressed by a mathematical formula. If x is the *independent* variable and y is the *dependent* variable, a functional relation is of the form

$$y = f(x)$$

Given a particular value of x, the function $f(x)$ gives the corresponding value of y. The observations, when plotted on a graph, all fall directly on the line or curve of the functional relationship. This is the main characteristic of all functional relationships.

14.1.2 Statistical Relation between Two Variables. A *statistical relation* is a relation where the value of the dependent variable is not uniquely determined when the level of the independent variable is specified. A statistical relation, unlike a functional relation, is not exact. The value of y is not uniquely determined from knowledge of x. The observations, when plotted on a graph, do not fall directly on the line or curve of the relationship. This is the main characteristic of all statistical relationships.

In many fields such as business, economics and administration exact relations are not generally observed among the variables, but rather statistical relationships prevail. For example: (*i*) The grade point Y secured by a student in the college is undoubtedly related to his grade point x secured in the school. (*ii*) The consumption expenditure Y of a household is related to its income x. (*iii*) The maintenance cost Y per year for an automobile is related to his age x. (*iv*) The yield Y of wheat is related to the quantity x of a fertilizer. (*v*) The amount of sales Y of a newly produced item may be related to its advertising cost x. (*vi*) The weight Y of a baby is certainly related to his age x. (*vii*) The saving Y of a person or a firm is related to his/its income. (*viii*) The height Y of a son is undoubtedly related to the height x of his father, *etc*.

Causal Relation. Another factor to consider is whether a causal relationship exists between two variables. In the example of steel output and labour input, it is clear that a causal relationship does exist. The number of workers will influence the number of tons of steel produced. There is also a causal relationship between hours of sunshine and the rate of growth of tulips. Conversely, it is less clear that more steel output will cause a rise in the number of workers, nor will we make the sunshine more by forcing tulips to grow faster. It is important to note regression analysis and correlation analysis make no assertions about causality.